

CONFORMAL GEOMETRY OF marginally TRAPPED SURFACES IN \mathbb{S}_1^4

E. HULETT

ABSTRACT. A spacelike surface $S \subset \mathbb{S}_1^4$ is marginally trapped if its mean curvature vector is lightlike. On any oriented spacelike surface $S \subset \mathbb{S}_1^4$ we show that a choice of orientation of the normal bundle $\nu(S)$ determines a smooth map $G : S \rightarrow \mathbb{S}^3$ which we call the null Gauss map of S . We show that if S is marginally trapped then G is a conformal immersion away the zeros of certain quadratic Hopf-differential of S and so the surface $G(S)$ is uniquely determined up to conformal transformations of \mathbb{S}^3 by two invariants: the normal Hopf differential κ and the Schwartzian derivative s . We show that these invariants plus an additional quadratic differential δ are related by a differential equation and determine the geometry of S up to ambient isometries of \mathbb{S}_1^4 . This allows us to obtain a characterization of marginally trapped surfaces S whose null Gauss image is a *constrained Willmore* surface in \mathbb{S}^3 [6]. As an application of these results we construct and study integrable non-trivial one-parameter deformations of marginally trapped surfaces with non-zero parallel mean curvature vector and those with flat normal bundle.

1. INTRODUCTION

A spacelike surface immersed in a 4-dimensional Lorentz manifold is called marginally trapped if its mean curvature vector is everywhere null or lightlike. The notion of marginally trapped surfaces was introduced by R. Penrose and plays a key role in the singularity theory of Einstein's equations [14]. The marginally trapped equation $\langle \vec{H}, \vec{H} \rangle = 0$ is interpreted in relativity theory as the condition describing the event horizon of a black hole [14]. In differential geometry marginally trapped surfaces are viewed as natural generalizations of minimal surfaces.

Different aspects of the geometry of marginally trapped surfaces have drawn the attention of geometers recently. In [1], [23] the authors provide different Weierstrass-type representation formulas of marginally trapped surfaces in \mathbb{R}_1^4 . Also [13] and [14] deal with a classifications of marginally trapped surfaces with parallel mean curvature. The notion of marginally trappedness has also been considered recently in higher dimensions and co-dimensions with very interesting results, see [3], [4].

Our goal in this paper is to study geometric properties of oriented marginally trapped surfaces in \mathbb{S}_1^4 in terms of conformal invariants of these surfaces. As applications of these ideas we construct spectral non-trivial deformations of marginally trapped surfaces with parallel non-zero mean curvature and those with flat normal bundle.

More specifically, given an oriented spacelike surface $S \subset \mathbb{S}_1^4$ its normal bundle $\nu(S)$ is Lorentzian hence at each point x of S there are two linearly independent null directions say, $n_+(x), n_-(x)$ which vary smoothly with x and determine a pair of smooth maps from S to the 3-sphere \mathbb{S}^3 , viewed as the manifold of null directions of Minkowski space \mathbb{R}_1^5 . Such maps can be interpreted as (pseudo) inverses of the conformal Gauss map Y introduced by R. L. Bryant [8]. We show that a choice of orientation on $\nu(S)$ distinguishes a preferred map, say n_+ which we call the *null Gauss map* G of the spacelike surface S . When S is marginally trapped and certain Hopf quadratic differential is never zero on S , then G is a conformal immersion of the surface S into the conformal sphere \mathbb{S}^3 and its geometry is dictated by two conformal invariants: the Schwartzian s and the normal Hopf differential κ . These invariants were introduced and studied by Burstall et al. in [11], see also [32].

2010 *Mathematics Subject Classification.* 53C42, 53C50, 53C43.

Key words and phrases. marginally trapped surfaces, null Gauss map, conformal invariants, harmonic map, integrable deformations, schwartzian, associated families.

Partially supported by research grants from CONICET, SECYT-UNC and FONCyT Argentina.

In Section 4 we obtain an equation which relates the conformal invariants s, κ with the δ -quadratic differential, a new geometric invariant of the corresponding marginally trapped surface. As a first consequence of this equation we prove Theorem 4.6 which says that the null Gauss map of an oriented marginally trapped surface S is a constrained Willmore surface in \mathbb{S}^3 if and only if S has non-zero parallel mean curvature vector. Constrained Willmore surfaces were introduced and studied in [6]. They are defined as extremes of the Willmore energy with respect to variations preserving the underlying conformal structure of the surface. A second consequence is Theorem 4.3 which states that a marginally trapped surface is essentially determined up to ambient isometries by the conformal invariants κ, s of its null Gauss map.

In Section 2 we fix notations and derive the structure equations of spacelike surfaces in \mathbb{S}_1^4 . Section 3 contains a short survey of $O(3,1)$ -invariant geometry of surfaces in the conformal sphere \mathbb{S}^3 . For a detailed exposition on the conformal invariant geometry of surfaces in \mathbb{S}^n see [11], [21] and [32].

In Section 5 we consider marginally trapped surfaces admitting non-trivial integrable one parameter deformations. The deformation is induced by a spectral parameter which determine symmetries of the compatibility equations thus giving rise to one-parameter families of surfaces obtained by deformation of a given surface. We consider here deformations of two kinds of marginally trapped surfaces in \mathbb{S}_1^4 namely, surfaces with non-zero parallel mean curvature vector, and surfaces with flat normal bundle. In the first case we show that the deformation originates in the associated family of an auxiliar harmonic map ϕ with values in a pseudo riemannian complex quadric Q . We show that a marginally trapped surface $f : \Sigma \rightarrow \mathbb{S}_1^4$ has constrained Willmore null Gauss map if and only if an auxiliar map ϕ with values in the complex quadric Q is harmonic. We use the associated family of ϕ to obtain a symmetry of the compatibility equations of the constrained Willmore null Gauss map of f , thus giving rise to the associated family of f and its null Gauss map. For marginally trapped surfaces with flat normal bundle we obtain a one-parameter deformation which originates in the so-called *Calapso-Bianchi* or isothermic T-transformation of isothermic surfaces in \mathbb{S}^3 [11], [32]. Motivated by [11] we show that both deformations may be unified in an extended action of $\mathbb{C} - \{0\}$ on the class of marginally trapped surfaces with non-zero parallel mean curvature. We conclude with a description of this extended action on non-isotropic marginally trapped tori with non-zero parallel mean curvature vector.

2. PRELIMINARIES

Denote by \mathbb{R}_1^5 the real 5-dimensional vector space with canonical coordinates $(x_0, x_1, x_2, x_3, x_4)$ equipped with the Lorentz inner product

$$(1) \quad \langle x, y \rangle = x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4.$$

De Sitter 4-space is defined as the unit sphere in \mathbb{R}_1^5 :

$$\mathbb{S}_1^4 = \{x \in \mathbb{R}_1^5 : \langle x, x \rangle = 1\}$$

Thus \mathbb{S}_1^4 is a connected simply connected 4-dimensional manifold which inherits from \mathbb{R}_1^5 a lorentzian metric $\langle \cdot, \cdot \rangle$ of constant sectional curvature $+1$. The complex bilinear extension of the Lorentz metric to \mathbb{C}^5 is given by

$$\langle z, w \rangle = z_0 w_0 + z_1 w_1 + z_2 w_2 + z_3 w_3 - z_4 w_4$$

and the corresponding (pseudo) hermitian inner product is given by $\langle z, \bar{w} \rangle$. We denote by \mathbb{C}_1^5 the complex space \mathbb{C}^5 endowed with the inner product $\langle z, \bar{w} \rangle$.

The Lie group $SO(4,1)$ acts transitively on \mathbb{S}_1^4 by isometries, so that choosing $e_0 \in \mathbb{S}_1^4$ as the base point, then \mathbb{S}_1^4 is isometric to the (pseudo) riemannian symmetric space $SO(4,1)/SO(3,1)$.

A non-zero vector $X \in \mathbb{R}_1^5$ is said to be *future pointing* if $\langle X, e_4 \rangle < 0$. This induces a time orientation on \mathbb{S}_1^4 : a non-zero tangent vector $X \in T_p \mathbb{S}_1^4$ is future pointing if its translated to the origin is future pointing. If X is future pointing and satisfies $\langle X, X \rangle = -1$, then (its translated) X lies in the real 4-hyperbolic space $\mathbb{H}^4 = \{x \in \mathbb{R}_1^5 : \langle x, x \rangle = -1, x_4 > 0\}$.

Let Σ be a connected orientable surface and $f : \Sigma \rightarrow \mathbb{S}_1^4$ a spacelike immersion i.e. the induced metric $g = f^*\langle \cdot, \cdot \rangle$ is Riemannian and it determines a conformal structure on Σ . Then f preserves this conformal structure i.e. $\langle f_z, f_z \rangle = 0$, for every local complex coordinate $z = x + iy$ on Σ , where $\partial_z = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$, and $\partial_{\bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$, are the complex partial operators. Equivalently,

$$(2) \quad \langle f_x, f_y \rangle = 0, \quad \|f_x\|^2 = \|f_y\|^2 > 0.$$

Conversely, if $f : \Sigma \rightarrow \mathbb{S}_1^4$ is a conformal immersion from a Riemann surface, then $\langle f_x, f_y \rangle = 0$, and $\|f_x\|^2 = \|f_y\|^2 \neq 0$, for every local complex coordinate $z = x + iy$. Since the ambient \mathbb{S}_1^4 is lorentzian, f_x, f_y have positive squared norm $\|f_x\|^2 = \|f_y\|^2 > 0$, and so $f : (\Sigma, g) \rightarrow \mathbb{S}_1^4$ is a spacelike isometric immersion, where g is the induced metric. Respect to a local complex coordinate $z = x + iy$ on Σ we introduce a conformal parameter u by $\langle f_z, f_{\bar{z}} \rangle = e^{2u}$, so that $g = 2e^{2u}(dx^2 + dy^2)$ is the local expression of the induced metric. Since f is conformal we have $2\langle f_{\bar{z}z}, f_z \rangle = \partial_z \langle f_z, f_z \rangle = 0$ and also $2\langle f_{\bar{z}z}, f_{\bar{z}} \rangle = \partial_z \langle f_{\bar{z}}, f_{\bar{z}} \rangle = 0$, which says that $f_{\bar{z}z}$ has no tangential component.

The second fundamental form of f is defined by $\langle \mathbb{I}(X, Y), N \rangle = -\langle df(X), dN(Y) \rangle$, for $X, Y \in T\Sigma$ and for every normal field N along f . The mean curvature vector of f is the trace of \mathbb{I} : $\vec{H} := \frac{1}{2} \text{trace} \mathbb{I}$. Since $f_{\bar{z}z}$ has no tangential component it decomposes into its f and \vec{H} components by $f_{\bar{z}z} = -e^{2u}f + e^{2u}\vec{H}$.

The pullback bundle of the tangent bundle of \mathbb{S}_1^4 decomposes into the tangent bundle and the normal bundle of f : $f^*(T\mathbb{S}_1^4) = T\Sigma \oplus \nu(f)$. Since $f : \Sigma \rightarrow \mathbb{S}_1^4$ is spacelike and Σ is orientable, the normal bundle $\nu(f)$ is an orientable lorentzian vector bundle. Fixing an orientation on $\nu(f)$, let $\{N_1, N_2\} \subset \Gamma(\nu(f))$ be an (ordered) orthonormal frame satisfying

$$\langle N_2, N_2 \rangle = -1, \quad \langle N_1, N_2 \rangle = 0, \quad \langle N_1, N_1 \rangle = 1.$$

If we demand that N_2 be future pointing, then either $\{N_1, N_2\}$ has the same orientation as $\nu(f)$, or $\{-N_1, N_2\}$ has the same orientation as $\nu(f)$. We say that an orthonormal frame $\{N_1, N_2\} \subset \Gamma(\nu(f))$ is *positively oriented*, $\{N_1, N_2\}$ has the same orientation as $\nu(f)$ and N_2 is (timelike) future pointing. Note that if $\{N_1, N_2\}$ is positively oriented then $\{-N_1, -N_2\}$ has the same orientation as $\nu(f)$, but it is not positively oriented since $-N_2$ points to the past.

In terms of a normal orthonormal frame the second fundamental form is given by

$$(3) \quad \mathbb{I} = -\langle df, dN_1 \rangle N_1 + \langle df, dN_2 \rangle N_2.$$

Let $\xi_1 := \langle f_{zz}, N_1 \rangle$, $\xi_2 := -\langle f_{zz}, N_2 \rangle$. Since f is conformal an easy calculation gives $f_{zz} = 2u_z f_z + \xi_1 N_1 + \xi_2 N_2$. In particular the $(2, 0)$ -part of \mathbb{I} is given by

$$(4) \quad \mathbb{I}(\partial_z, \partial_z) = \xi_1 N_1 + \xi_2 N_2$$

Set $h_1 := \langle \vec{H}, N_1 \rangle$ and $h_2 := -\langle \vec{H}, N_2 \rangle$, then $\vec{H} = h_1 N_1 + h_2 N_2$.

The structure equations of a conformal immersion $f : \Sigma \rightarrow \mathbb{S}_1^4$ are given by

$$(5) \quad \begin{aligned} f_{zz} &= 2u_z f_z + \xi_1 N_1 + \xi_2 N_2 \\ f_{\bar{z}z} &= -e^{2u}f + e^{2u}\vec{H}, \\ \partial_z N_1 &= -h_1 f_z - e^{-2u}\xi_1 f_{\bar{z}} + \sigma N_2, \\ \partial_z N_2 &= h_2 f_z + e^{-2u}\xi_2 f_{\bar{z}} + \sigma N_1, \end{aligned}$$

and the compatibility among these equations is just Gauss's, Codazzi's and Ricci's equation:

$$(6) \quad \begin{aligned} \text{Gauss,} \quad & 2u_{\bar{z}z} = -e^{2u} + e^{-2u}(|\xi_1|^2 - |\xi_2|^2) - e^{2u}\|\vec{H}\|^2, \\ \text{Codazzi,} \quad & e^{2u}(\partial_z h_1 + \sigma h_2) = \partial_{\bar{z}}\xi_1 + \xi_2 \bar{\sigma}, \\ & e^{2u}(\partial_z h_2 + \sigma h_1) = \partial_{\bar{z}}\xi_2 + \xi_1 \bar{\sigma}, \\ \text{Ricci,} \quad & \text{Im}(\sigma_{\bar{z}}) = e^{-2u}\text{Im}(\xi_1 \xi_2). \end{aligned}$$

A spacelike surface $f : \Sigma \rightarrow \mathbb{S}_1^4$ is called *marginally trapped* if its mean curvature vector is null or lightlike: $\langle \vec{H}, \vec{H} \rangle = 0$. If $\vec{H} \neq 0$, then after a change of orientation of the normal bundle (i.e. after

a change of sign $N_1 \mapsto -N_1$) if necessary, the marginally trapped condition $\langle \vec{H}, \vec{H} \rangle = h_1^2 - h_2^2 = 0$, reads $h_1 = h_2$, with $h_1 + h_2 \neq 0$, in this case the mean curvature vector satisfies

$$(7) \quad \vec{H} = h(N_1 + N_2), \quad \text{with } h = h_1 = h_2.$$

We call h the *mean curvature function* of f respect to the positively oriented lorentzian normal frame $\{N_1, N_2\}$. From the second structure equation $f_{\bar{z}z} = -e^{2u}f + e^{2u}\vec{H}$, it follows that the mean curvature function h satisfies

$$(8) \quad h = e^{-2u}\langle f_{\bar{z}z}, N_1 \rangle = -e^{-2u}\langle f_{\bar{z}z}, N_2 \rangle.$$

Hence f is marginally trapped iff $\langle f_{\bar{z}z}, (N_1 + N_2) \rangle = 0$. Introducing the function $\sigma := -\langle \partial_z N_1, N_2 \rangle$, the structure equations of a marginally trapped immersion f are thus given by

$$(9) \quad \begin{aligned} f_{zz} &= 2u_z f_z + \xi_1 N_1 + \xi_2 N_2, \\ f_{\bar{z}z} &= -e^{2u}f + e^{2u}\vec{H}, \\ \partial_z N_1 &= -h f_z - e^{-2u}\xi_1 f_{\bar{z}} + \sigma N_2, \\ \partial_z N_2 &= h f_z + e^{-2u}\xi_2 f_{\bar{z}} + \sigma N_1, \end{aligned}$$

where $\{N_1, N_2\} \subset \Gamma(\nu(f))$ is a positively oriented orthonormal frame. The compatibility conditions or equations of Gauss, Codazzi and Ricci above reduce to:

$$(10) \quad \begin{aligned} 2u_{\bar{z}z} &= -e^{2u} + e^{-2u}(|\xi_1|^2 - |\xi_2|^2), \\ e^{-2u}(\xi_{1\bar{z}} + \xi_2 \bar{\sigma}) &= (h_z + \sigma h), \\ e^{-2u}(\xi_{2\bar{z}} + \xi_1 \bar{\sigma}) &= (h_z + \sigma h), \\ \text{Im}(\sigma_{\bar{z}}) &= e^{-2u} \text{Im}(\xi_1 \bar{\xi}_2). \end{aligned}$$

The Gaussian curvature of the induced metric g is given by $K = -\Delta_g u = -2e^{-2u}u_{\bar{z}z}$, where $\Delta_g = 2e^{-2u}\partial_{\bar{z}}\partial_z$, is the Laplace operator of the induced metric g . From Gauss equation (10) we obtain the expression of the Gaussian curvature of the induced metric on Σ ,

$$(11) \quad K = 1 - e^{-4u}(|\xi_1|^2 - |\xi_2|^2),$$

Let ∇^\perp denote the covariant derivative on the normal bundle $\nu(f)$, then $\omega := \langle \nabla^\perp N_2, N_1 \rangle$ is the corresponding connection one form. Fixed an orientation on the normal bundle $\nu(f)$ the normal curvature is defined by $d\omega = K^\perp dA_g$, where dA_g is the area form of the induced metric g . Thus $\omega = 2\text{Re}(\sigma dz)$, and so $d\omega = -4\text{Im}(\sigma_{\bar{z}})dx \wedge dy$. From Ricci equation above it follows that $\text{Im}(\sigma_{\bar{z}}) = e^{-2u}\text{Im}(\xi_1 \bar{\xi}_2)$. Thus since $dA_g = 2e^{2u}dx \wedge dy$, the normal curvature function is given by

$$(12) \quad K^\perp = -e^{-2u}\text{Im}(\sigma_{\bar{z}}) = -e^{-2u}\text{Im}(\xi_1 \bar{\xi}_2).$$

Then the normal bundle is flat if and only if $K^\perp = 0$.

On the other hand from Codazzi's equation the covariant derivative of the mean curvature vector of a conformal immersion $f : \Sigma \rightarrow \mathbb{S}_1^4$ is given by

$$(13) \quad \nabla_{\partial_z}^\perp \vec{H} = (\partial_z h_1 + \sigma h_2)N_1 + (\partial_z h_2 + \sigma h_1)N_2.$$

In particular if f is marginally trapped then in a positively oriented orthonormal frame $\{N_1, N_2\} \subset \Gamma(\nu(f))$ the above formula becomes

$$(14) \quad \nabla_{\partial_z}^\perp \vec{H} = e^{-2u}(h_z + \sigma h)(N_1 + N_2).$$

Hence f has parallel mean curvature vector if and only if $h_z + \sigma h = 0$.

3. SURFACE THEORY IN THE CONFORMAL SPHERE \mathbb{S}^3

We give a brief account of Moebius surface geometry in \mathbb{S}^3 such as exposed in [11]. For detailed proofs and further developements we refer the reader to [11], [21] and [32].

The null or light cone in \mathbb{R}_1^5 is defined by

$$(15) \quad \mathcal{L} = \{0 \neq x \in \mathbb{R}_1^5 : \langle x, x \rangle = 0\}.$$

The *future light cone* $\mathcal{L}_+ \subset \mathcal{L}$ consists of future pointing vectors $x \in \mathcal{L}$. For every $x \in \mathbb{S}^3 \subset \mathbb{R}^4$, the point $(x, 1) \in \mathbb{R}_1^5$ lies in the future light cone \mathcal{L}_+ . We are using here the fact that any vector x in \mathbb{R}_1^5 may be uniquely written as an ordered pair (x', t) with $x' \in \mathbb{R}^4$ and $t \in \mathbb{R}$, thus giving rise to an isomorphism $\mathbb{R}^4 \oplus \mathbb{R} \rightarrow \mathbb{R}_1^5$. In particular points on \mathcal{L} are of the form $(x, \pm \|x\|^2)$, with $x \in \mathbb{R}^4$. The map $\mathbb{S}^3 \ni x \mapsto [(x, 1)]$ identifies the unit round sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ with the projectivization of the light cone, $P(\mathcal{L}) \subset \mathbb{RP}^5$. Let $O_+(4, 1)$ be the group of orthogonal transformations of \mathbb{R}_1^5 preserving the time orientation. Then each $F \in O_+(4, 1)$ maps null lines to null lines hence preserves the light cone \mathcal{L} . Moreover it is easy to see that $O_+(4, 1)$ acts transitively on \mathbb{S}^3 by $g.[x] = [gx]$. Here $O_+(4, 1)$ is referred to as the group of Moebius transformations of the conformal sphere \mathbb{S}^3 . Note that the subgroup of $O(4, 1)$ preserving $P(\mathcal{L}_+)$, is precisely $O_+(4, 1)$.

A smooth map into the conformal sphere $\psi : \Sigma \rightarrow \mathbb{S}^3 \equiv P(\mathcal{L})$ can be viewed as a null line subbundle Λ of the trivial bundle $\Sigma \times \mathbb{R}_1^5$ via $\psi(x) = \Lambda_x$, $x \in \Sigma$. A (local) lift of ψ is a smooth map $X : U \rightarrow \mathcal{L}$ from an open subset $U \subset \Sigma$, such that the null line spanned by $X(x)$ is Λ_x for every $x \in U$. The map ψ is called a *conformal immersion* if every local lift X of ψ is conformal, i.e. $\langle X_z, X_z \rangle = 0$, $\langle X_z, X_{\bar{z}} \rangle > 0$, for every coordinate z .

Let $V := \text{span}\{X, dX, X_{z\bar{z}}\}$, where X is a conformal lift of ψ . It is easily seen that V is in fact independent on the election of a local coordinate z and any particular conformal lift of ψ . So V can be viewed as a vector sub-bundle $V \subset \mathbb{R}_1^5 \times \Sigma$ on which the ambient metric of \mathbb{R}_1^5 induces a vector bundle metric of signature $(3, 1)$. Each fiber V_x determines a Moebius invariant 2-sphere $\mathbb{S}^2(x) \equiv P(V_x \cap \mathcal{L}) \subset P(\mathcal{L}) \cong \mathbb{S}^3$. These spheres altogether comprise the so-called *mean curvature sphere* or *central sphere congruence* of the surface ψ [11].

Respect to a fixed local coordinate $z : U \rightarrow \mathbb{C}$ there is a distinguished local lift $Y : U \rightarrow \mathcal{L}_+$ of ψ taking values in the future light cone such that

$$\langle Y_z, Y_{\bar{z}} \rangle = \frac{1}{2},$$

or equivalently $|dY|^2 = |dz|^2$ on U . It is called the *canonical lift* of the surface ψ and is Moebius invariant.

The complementary orthogonal line sub-bundle V^\perp is determined by $\Sigma \times \mathbb{R}_1^5 = V \oplus V^\perp$ and the connection D on V^\perp is just orthogonal projection of the usual derivative in \mathbb{R}_1^5 :

$$D_X v = [d_X v]^\perp, \quad v \in \Gamma(V^\perp), \quad X \in T\Sigma.$$

Let $N \in \Gamma(V)$ be the unique section satisfying

$$\langle N, N \rangle = \langle N, Y_z \rangle = \langle N, Y_{\bar{z}} \rangle = 0, \quad \langle Y, N \rangle = -1.$$

Thus $V = \text{span}\{Y, \text{Re}(Y_z), \text{Im}(Y_z), N\}$ and it is shown in [11] that the Moebius invariant frame $\{Y, Y_z, Y_{\bar{z}}, N\} \subset \Gamma(V \otimes \mathbb{C})$ satisfies orthogonally relations given by

$$(16) \quad \begin{aligned} \langle Y, Y \rangle &= \langle N, N \rangle = 0, & \langle N, Y \rangle &= -1, \\ \langle Y, dY \rangle &= \langle N, dY \rangle = \langle dN, N \rangle = 0, \\ \langle Y_z, Y_z \rangle &= \langle Y_{\bar{z}}, Y_{\bar{z}} \rangle = 0, & \langle Y_z, Y_{\bar{z}} \rangle &= \frac{1}{2}. \end{aligned}$$

A direct consequence of the above equations is that Y_{zz} is orthogonal to Y, Y_z and $Y_{\bar{z}}$ and so there is a unique election of a local complex function s on Σ for which $Y_{zz} + \frac{s}{2}Y$ is a section of the normal bundle $V^\perp \otimes \mathbb{C}$ namely, $\frac{s}{2} = \langle Y_{zz}, N \rangle$. Thus we arrive at the fundamental equation of Moebius invariant surface geometry:

$$(17) \quad Y_{zz} + \frac{s}{2}Y = \kappa,$$

defining uniquely the complex valued function s and the section κ of $V^\perp \otimes \mathbb{C}$, respect to the local coordinate z . The function s is interpreted as the *schwartzian derivative* of the conformal immersion ψ with respect to z , and κ is identified with the *normal valued Hopf differential* of ψ , respect to the coordinate z . By construction s and κ are Moebius invariants of the immersion ψ with respect to a given coordinate z .

In [11] there is an interpretation of κ in terms of euclidean invariants of the immersion ψ which we briefly describe: There is a unique conformal immersion $\widehat{\psi} : \Sigma \rightarrow \mathbb{S}^3 \subset \mathbb{R}^4$ satisfying $[(\widehat{\psi}(x), 1)] = \psi(x)$, $\forall x \in \Sigma$. Thus $\phi = (\widehat{\psi}(x), 1)$ is a lift of ψ , which is called the euclidean lift of ψ [11]. Let $\nu(\widehat{\psi})$ denote the normal bundle of the immersed surface $\widehat{\psi}$. Then there is a bundle isomorphism $\nu(\widehat{\psi}) \cong V^\perp$ given by

$$(18) \quad v \mapsto \langle v, \widehat{H} \rangle (\widehat{\psi}, 1) + (v, 0).$$

where \widehat{H} is the mean curvature vector of $\widehat{\psi}$. Under this isomorphism $\kappa \in \Gamma(V^\perp \otimes \mathbb{C})$ corresponds to a complex section $\widehat{\kappa} \in \nu(\widehat{\psi}) \otimes \mathbb{C}$ satisfying $\kappa = \langle \widehat{\kappa}, \widehat{H} \rangle (\widehat{\psi}, 1) + (\widehat{\kappa}, 0)$. Using (17) it is shown that

$$\widehat{\kappa} \frac{dz^2}{|dz|} = \frac{\mathbb{I}^{(2,0)}}{|d\phi|},$$

where $\mathbb{I}^{(2,0)}$ is the $(2,0)$ -part of the normal bundle valued (euclidean) second fundamental form of $\widehat{\psi}$. In this way κ , up to the isomorphism (18), is the trace free part of the second fundamental form, i.e., the normal bundle valued Hopf differential of $\widehat{\psi}$, scaled by the square root of the $\widehat{\psi}$ -induced metric.

The following structural equations of a conformal immersion $\psi : \Sigma \rightarrow \mathbb{S}^3$ were obtained in [11] from the above orthogonality conditions:

$$(19) \quad \begin{aligned} (i) \quad & Y_{zz} = -\frac{s}{2}Y + \kappa, \\ (ii) \quad & Y_{\bar{z}\bar{z}} = -\langle \kappa, \bar{\kappa} \rangle Y + \frac{1}{2}N, \\ (iii) \quad & N_z = -2\langle \kappa, \bar{\kappa} \rangle Y_z - sY_{\bar{z}} + 2D_{\bar{z}}\kappa. \end{aligned}$$

The compatibility among these are the following equations,

$$(20) \quad \begin{aligned} \text{Conformal Gauss:} \quad & \frac{s}{2} = 3\langle \bar{\kappa}_z, \kappa \rangle + \langle \bar{\kappa}, \kappa_z \rangle, \\ \text{Conformal Codazzi:} \quad & \text{Im}(\kappa_{\bar{z}\bar{z}} + \frac{s}{2}\kappa) = 0. \end{aligned}$$

Also when the local coordinate changes from z to w the new invariants s' and κ' change according to

$$(21) \quad \begin{aligned} \kappa' &= \kappa \left(\frac{\partial z}{\partial w} \right)^{\frac{3}{2}} \left(\frac{\partial \bar{z}}{\partial \bar{w}} \right)^{-\frac{1}{2}}, \\ s' &= s \left(\frac{\partial z}{\partial w} \right)^2 + S_w(z), \end{aligned}$$

where the usual Schwartzian derivative of a meromorphic function $g : \Sigma \rightarrow \mathbb{C}$ is given by $S_z(g) = (\frac{g''}{g'})' - \frac{1}{2}(\frac{g''}{g'})^2$. The importance of the conformal Gauss and Codazzi's equations is reflected in the following fundamental theorem of conformal surface theory,

Theorem 3.1. [11] *Let Σ be a Riemann surface and $\psi_j : \Sigma \rightarrow \mathbb{S}^3$ be conformal immersed surfaces inducing the same Hopf differentials and the same Schwartzians. Then there is a Moebius transformation $T : \mathbb{S}^3 \rightarrow \mathbb{S}^3$ with $T\psi_1 = \psi_2$.*

Conversely, let κ and s be given data on Σ transforming according (21), which also satisfy the conformal Gauss and Codazzi equations (20). Then there exists a conformal immersion $x : \Sigma \rightarrow \mathbb{S}^3$ with Hopf differential κ and Schwartzian s .

Remark 3.1. *It is proved in [11] that $\kappa \frac{dz^2}{|dz|}$ is a globally defined quadratic differential with values in $L \otimes \mathbb{C}$, where L is the real line bundle $(K \otimes \bar{K})^{1/2}$ of densities of conformal weight 1 over Σ [12]. Then for any local coordinate system (U, z) , κ can be viewed just as a local complex function on $U \subset \Sigma$ which transforms according (21).*

Remark 3.2. *If a conformal immersion $\psi : \Sigma \rightarrow \mathbb{S}^3$ has $\kappa \equiv 0$, then the image of ψ is contained in a fixed 2-sphere $\mathbb{S}^2 \subset \mathbb{S}^3$, as follows from (19). Considering ψ as a conformal map $\psi : \Sigma \rightarrow \mathbb{S}^2 \equiv \mathbb{CP}^1$, it is shown in [11] that $s = (\frac{\psi''}{\psi'})' - \frac{1}{2}(\frac{\psi''}{\psi'})^2$ which is the usual Schwartzian derivative of ψ . In this case it is shown that s uniquely determines ψ up to transformations of $PSl(2, \mathbb{C})$, the Moebius transformation group of \mathbb{CP}^1 .*

The map $\gamma : \Sigma \ni x \mapsto V(x)$ with values in the Grassmannian $G_{3,1}(\mathbb{R}^5)$ is called *the conformal Gauss map* of the immersion ψ [8], [11], [17], [24] [32]. γ induces a positive definite conformal metric on Σ given by $g_\gamma = \frac{1}{4}\langle d\gamma, d\gamma \rangle = |\kappa|^2 |dz|^2$ [32]. The Willmore energy of the conformal immersion ψ is defined as the total area of (Σ, g_γ) and is given by

$$(22) \quad W(\psi) = \frac{i}{2} \int_{\Sigma} |\kappa|^2 dz \wedge d\bar{z},$$

which coincides (up to a constant multiple) with the Willmore energy of the immersion ψ [11]. A conformal immersion $\psi : \Sigma \rightarrow \mathbb{S}^3$ is called a *Willmore surface* if it extremizes the Willmore energy functional (22). It is known [11] that ψ is Willmore iff its conformal invariants κ and s (the Hopf differential and the Schwartzian derivative) satisfy the following stronger version of the conformal Codazzi's equation:

$$(23) \quad \kappa_{\bar{z}\bar{z}} + \frac{\bar{s}}{2}\kappa = 0.$$

Spacelike surfaces in \mathbb{S}_1^4 are related to surfaces in \mathbb{S}^3 and \mathbb{R}^3 through a double cover of the conformal Gauss map γ , which is the Bryant's Gauss conformal map Y : given an oriented surface $\psi : \Sigma \rightarrow \mathbb{S}^3$ with mean curvature H , the conformal Gauss map Y_ψ assigns to a point $x \in \Sigma$ the oriented sphere $S(x) \subset \mathbb{S}^3$ of radius $|H(x)|^{-1}$ in contact with the surface at $\psi(x)$. Thus Y_ψ takes values in the manifold of all oriented 2-spheres (and planes) in \mathbb{S}^3 which is identified with De Sitter 4-space \mathbb{S}_1^4 [21]. B. Palmer [25] observed that when Y_ψ has non-zero mean curvature vector it is marginally trapped. This fact is also implicit in the work of Blaschke [7].

4. THE NULL GAUSS MAP AND ITS CONFORMAL INVARIANTS

Let $f : \Sigma \rightarrow \mathbb{S}_1^4$ be a conformal (spacelike) immersion and fix an orientation on the normal bundle $\nu(f)$. Let $\{N_1, N_2\} \subset \Gamma(\nu(f))$ be a positively oriented lorentzian orthonormal frame. Then for each $p \in \Sigma$ the frame $\{N_1, N_2\}$ determines the null line $\text{span}\{N_1(p) + N_2(p)\}$. We claim that this null line depends only on p and not on $\{N_1, N_2\}$. In fact, if $\{N'_1, N'_2\} \subset \Gamma(\nu(f))$ is another positively oriented orthonormal frame then both frames are related by a gauge,

$$\begin{aligned} N'_1 &= \cosh(s)N_1 + \sinh(s)N_2, \\ N'_2 &= \sinh(s)N_1 + \cosh(s)N_2. \end{aligned}$$

from these equations it follows that $N'_1 + N'_2 = e^s(N_1 + N_2)$, and so $N'_1 + N'_2$ and $N_1 + N_2$ generate the same null line. Let

$$(24) \quad G : \Sigma \rightarrow \mathbb{S}^3, \quad G(x) = [N_1(x) + N_2(x)], \quad x \in \Sigma.$$

i.e. $G(x)$ is the null line generated by $N_1(x) + N_2(x)$, where $\{N_1, N_2\} \subset \Gamma(\nu(f))$ is any positively oriented orthonormal frame. Thus G is well defined by our previous observation and we call it *the null Gauss map of f* .

Denote by \widehat{G} the unique smooth map from Σ to the round euclidean sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ such that $[(\widehat{G}(x), 1)] = G(x)$ for every $x \in \Sigma$, then $\phi =: (\widehat{G}, 1) : \Sigma \rightarrow \mathcal{L}_+$ is called *the euclidean lift of G* . Thus $\phi =: (\widehat{G}, 1)$ takes values in the conic section $\mathcal{S} = \{x \in \mathcal{L} : \langle x, e_4 \rangle = -1\}$ which inherits from the ambient \mathbb{R}_1^5 a positive definite metric of constant curvature $+1$, and so it is a copy of the round 3-sphere of radius one.

For any positively oriented orthonormal frame $\{N_1, N_2\} \subset \Gamma(\nu(f))$, $X = N_1 + N_2$ is a local lift of G with values in \mathcal{L}_+ . Using the structure equations (9) we see that

$$(25) \quad X_z = N_{1,z} + N_{2,z} = (h_2 - h_1)f_z + e^{-2u}(\xi_2 - \xi_1)f_{\bar{z}} + \sigma X.$$

Hence $\langle X, f_z \rangle = \langle X, f_{\bar{z}} \rangle = 0$. Moreover since $\langle f_z, f_z \rangle = \langle f_{\bar{z}}, f_{\bar{z}} \rangle = 0$, $\langle f_z, f_{\bar{z}} \rangle = e^{2u}$, then

$$(26) \quad \langle X_z, X_z \rangle = \langle N_{1,z} + N_{2,z}, N_{1,z} + N_{2,z} \rangle = (h_2 - h_1)(\xi_2 - \xi_1)$$

Thus if f is marginally trapped $\langle X_z, X_z \rangle = 0$. Let Z be another local lift of G , then $X = \lambda Z$ for some smooth non-zero function λ . Respect to a local coordinate z we compute $X_z = \lambda_z Z + \lambda Z_z$.

Since $\langle Z, Z \rangle = 0$, then $0 = \langle Z, Z_z \rangle$, hence $0 = \langle X_z, X_z \rangle = \lambda^2 \langle Z_z, Z_z \rangle$, from which $\langle Z_z, Z_z \rangle = 0$ follows. On the other hand since $X_{\bar{z}} = \lambda_{\bar{z}} Z + \lambda Z_{\bar{z}}$, then from $\langle Z, Z_{\bar{z}} \rangle = 0$, and (25) we obtain

$$(27) \quad \lambda^2 \langle Z_z, Z_{\bar{z}} \rangle = e^{-2u} |\xi_1 - \xi_2|^2 = \langle X_z, X_{\bar{z}} \rangle.$$

Hence away from the zeros of $\xi_1 - \xi_2$ it follows that $\langle X_z, X_{\bar{z}} \rangle > 0$ and $\langle Z_z, Z_{\bar{z}} \rangle > 0$. In particular if $\xi_1 - \xi_2$ is never zero on Σ then $G : \Sigma \rightarrow \mathbb{S}^3$ is a conformal immersion.

We call $q := (\xi_1 - \xi_2)dz^2$ the Hopf quadratic differential of the marginally trapped surface $f : \Sigma \rightarrow \mathbb{S}_1^4$. The quadratic Hopf differential was introduced in [1] for marginally trapped surfaces in \mathbb{R}_1^4 . We have proved the following Lemma:

Lemma 4.1. *Let $f : \Sigma \rightarrow \mathbb{S}_1^4$ be a conformally immersed marginally trapped surface and q its quadratic Hopf differential. Then every (local) lift Z of the null Gauss map G satisfies $\langle Z_z, Z_z \rangle = 0$ and $\langle Z_z, Z_{\bar{z}} \rangle > 0$ away the zeros of q . In particular if $q(x) \neq 0, \forall x \in \Sigma$ then $G : \Sigma \rightarrow \mathbb{S}^3$ is a conformal immersion.*

Since $(\widehat{G}, 1)$ is a lift of G , then away the zeros of q , \widehat{G} satisfies $\langle \widehat{G}_z, \widehat{G}_z \rangle = 0$ and $\langle \widehat{G}_z, \widehat{G}_{\bar{z}} \rangle > 0$, where $\langle \cdot, \cdot \rangle$ is the round metric on the sphere \mathbb{S}^3 . Thus if q is never zero \widehat{G} is a conformal immersion into the round 3-sphere.

Let $f : \Sigma \rightarrow \mathbb{S}_1^4$ be a spacelike immersion then from Ricci's equation $\nu(f)$ is flat if and only if $Im(\sigma_{\bar{z}}) = 0$. In this case $\sigma_{\bar{z}} - \overline{\sigma_z} = \sigma_{\bar{z}} - \overline{\sigma_z} = 0$ which shows that the real one form $\eta := \sigma dz + \overline{\sigma} d\bar{z}$ is closed. Hence there is a locally defined smooth real function β such that $d\beta = \eta$. One can define a new positively oriented orthonormal lorentzian frame $\{N'_1, N'_2\}$ by

$$N'_1 = \cosh(\beta)N_1 + \sinh(\beta)N_2, \quad N'_2 = \sinh(\beta)N_1 + \cosh(\beta)N_2.$$

Then it is easy to check that the new frame $\{N'_1, N'_2\}$ has structure function $\sigma' = 0$, so that $\{N'_1, N'_2\}$ is a ∇^\perp -parallel frame which is unique up to (constant) hyperbolic rotations in $\nu(f)$. We keep denoting by $\{N_1, N_2\}$ this new positively oriented ∇^\perp -parallel orthonormal frame. If f is marginally trapped then Codazzi's equations (10) reduce to

$$(28) \quad \xi_{1,\bar{z}} = \xi_{2,\bar{z}} = e^{2u} h_z, \quad h = h_1 = h_2,$$

which imply $(\xi_1 - \xi_2)_{\bar{z}} = e^{2u}(h - h)_z = 0$, hence q is holomorphic. Conversely, if q is holomorphic then again by Codazzi's equation we obtain $0 = (\xi_1 - \xi_2)_{\bar{z}} = \overline{\sigma}(\xi_1 - \xi_2)$. If q does not vanish identically then σ must be zero away the isolated zeros of q , thus $\sigma \equiv 0$ by continuity. We have proved the following,

Lemma 4.2. *Let $f : \Sigma \rightarrow \mathbb{S}_1^4$ be a marginally trapped surface. If f has flat normal bundle the Hopf differential $q = (\xi_2 - \xi_1)dz^2$ is holomorphic. Conversely, if q is holomorphic and non-identically zero, then f has flat normal bundle.*

Remark 4.1. i) *If a conformally immersed surface $f : \Sigma \rightarrow \mathbb{S}_1^4$ has zero mean curvature vector then its normal bundle is not necessarily flat. In this case the Hopf differential q is holomorphic as consequence of Codazzi's equations (10).*

ii) *If $f : \Sigma \rightarrow \mathbb{S}_1^4$ is marginally trapped with parallel mean curvature vector then $\nu(f)$ is flat [18] and so q is holomorphic by Lemma 4.2.*

iii) *From (7) the ∇^\perp -derivative of the mean curvature vector of a marginally trapped surface in a positively oriented normal frame is given by*

$$\nabla_{\partial_z}^\perp \vec{H} = (h_z + \sigma h)(N_1 + N_2).$$

Thus $\nabla^\perp \vec{H} = 0$ implies $\nu(f)$ is flat [18], hence h is constant in a positively oriented ∇^\perp -parallel frame $\{N_1, N_2\} \subset \Gamma(\nu(f))$. Conversely if $\nu(f)$ is flat, then $\sigma = 0$ for any ∇^\perp -parallel orthonormal frame $\{N_1, N_2\} \subset \Gamma(\nu(f))$.

Remark 4.2. *If $q \equiv 0$, then by (27) $N_1 + N_2$ is a constant null line for every oriented lorentzian frame $\{N_1, N_2\}$, hence the null Gauss map G is constant. Since $\langle f, N_1 + N_2 \rangle = 0$, the surface f has constant curvature $K = 1$ by (11) and lies in the degenerated hypersurface $M_0 \subset \mathbb{S}_1^4$, which is the intersection of the degenerate 4-plane $[N_1 + N_2]^\perp$ in \mathbb{R}_1^5 with \mathbb{S}_1^4 . For instance this is just*

the case of any marginally trapped surface $f : \mathbb{S}^2 \rightarrow \mathbb{S}_1^4$ with flat normal bundle. In fact since q is holomorphic on \mathbb{S}^2 , it must vanish.

4.1. Spacelike isothermic surfaces. The normal valued quadratic Hopf differential of a spacelike immersion $f : \Sigma \rightarrow \mathbb{S}_1^4$ is the $\Gamma(\nu(f)) \otimes \mathbb{C}$ -valued two-form

$$\Omega = \xi_1 N_1 dz^2 + \xi_2 N_2 dz^2,$$

defined in terms of an orthonormal frame $\{N_1, N_2\} \subset \Gamma(\nu(f))$, where ξ_1, ξ_2 are the coefficients of $\mathbb{I}(\partial_z, \partial_z)$, the $(2, 0)$ -component of the second fundamental form of f . The spacelike surface $f : \Sigma \rightarrow \mathbb{S}_1^4$ is called isothermic [26] if for each point $x \in \Sigma$ there is a coordinate z for which the normal valued Hopf differential Ω is real-valued. Note that from Ricci's equation (6) it follows that every isothermic spacelike immersion in \mathbb{S}_1^4 has flat normal bundle.

4.2. Non-isotropic spacelike surfaces. A conformally (hence spacelike) immersed surface $f : \Sigma \rightarrow \mathbb{S}_1^4$ is called *non-isotropic* if the quartic complex differential $Q = \langle f_{zz}, f_{zz} \rangle dz^4$ is never zero on Σ . The quartic complex differential Q was introduced in [8] in the context of the conformal Gauss map. In terms of an orthonormal frame $\{N_1, N_2\} \subset \Gamma(\nu(f))$, $Q = (\xi_1^2 - \xi_2^2) dz^4$, thus if f is non-isotropic then the Hopf differential $q = (\xi_1 - \xi_2) dz^2$ is never zero and so the null Gauss map $G : \Sigma \rightarrow \mathbb{S}^3$ is a conformal immersion. The notion of isotropy has an interpretation in terms of the *curvature hyperbola* which is the image of the unit circle on $T_p \Sigma$ under the second fundamental form of f :

$$\{\mathbb{I}_p(X, X) : X \in T_p \Sigma, \|X\|^2 = 1\} \subset T_p^\perp \Sigma$$

It is shown that f is non-isotropic if and only if the curvature hyperbola at each point of Σ is non-equilateral [18]. A conformal non-isotropic spacelike immersion $f : \Sigma \rightarrow \mathbb{S}_1^4$ with zero mean curvature vector is also called harmonic superconformal [22]. Hence non-isotropic marginally trapped surfaces can be viewed as natural generalizations of harmonic superconformal surfaces.

4.3. Sphere congruences. Let $f : \Sigma \rightarrow \mathbb{S}_1^4$ be a non-isotropic marginally trapped surface with null Gauss map G and consider the central sphere congruence of the surface $G : \Sigma \rightarrow \mathbb{S}^3$, given by the subbundle $V = \text{span}\{X, dX, X_{z\bar{z}}\} \subset \Sigma \times \mathbb{R}_1^5$, where $X : \Sigma \rightarrow \mathcal{L}_+$ is any local lift of G . Since \mathbb{S}_1^4 identifies with the manifold of oriented 2-spheres in \mathbb{S}^3 , the immersion f is associated to the 2-sphere congruence $\Sigma \ni x \mapsto S(x)$, where $S(x)$ is the 2-sphere obtained by projectivization of the intersection of the Minkowski vector subspace $f^\perp(x) \subset \mathbb{R}_1^5$ with the null cone \mathcal{L} :

$$S(x) = P(f^\perp(x) \cap \mathcal{L}) \subset \mathbb{S}^3.$$

Note that the antipodal surface $(-f)$ determines the same sphere congruence $x \mapsto S(x)$. We say that $S(x)$ is oriented if it is associated to f , and opposite oriented if it is associated to $-f$. We claim that $f^\perp = V$, i.e. both sphere congruences coincide. To prove the claim we use the local lift of G given by $X := N_1 + N_2 : U \rightarrow \mathcal{L}_+$, where $\{N_1, N_2\} \subset \Gamma(\nu(f))$ is a positively oriented orthonormal lorentzian frame. Thus $V = \text{span}\{X, \text{Re}(X_z), \text{Im}(X_z), X_{z\bar{z}}\}$. In particular $\langle X, f \rangle = 0$ since N_1, N_2 are normal to f . On the other hand from (25),

$$(29) \quad X_z = e^{-2u}(\xi_2 - \xi_1)f_{\bar{z}} + \sigma X.$$

Hence $\langle f, X_z \rangle = \langle f, X_{\bar{z}} \rangle = 0$, or $\langle f, dX \rangle = 0$. Since every lift W of G is a multiple of X by some function, then W satisfies $\langle f, W \rangle = 0$ and $\langle f, dW \rangle = 0$. This just says that G is an envelope of the congruence determined by f [21].

On the other hand taking $\partial_{\bar{z}}$ on (29) and using again (9) yields

$$X_{z\bar{z}} = e^{-2u}(\xi_2 - \xi_1)(\bar{\xi}_1 N_1 + \bar{\xi}_2 N_2) + \sigma e^{-2u}(\bar{\xi}_2 - \bar{\xi}_1)f_z + (\sigma_{\bar{z}} + |\sigma|^2)X,$$

from which $\langle f, X_{z\bar{z}} \rangle = 0$ follows and so $V \subseteq f^\perp$. Thus $V = f^\perp$ since V has rank four. We have proved the following

Proposition 4.1. *Let $f : \Sigma \rightarrow \mathbb{S}_1^4$ be a non-isotropic conformal marginally trapped immersion with null Gauss map $G : \Sigma \rightarrow \mathbb{S}^3$. Then G is an envelope of the spherical congruence determined by f . Moreover, the central sphere congruence of the null Gauss map G coincides with the spherical congruence determined by $\pm f$.*

Recall from Section 3 that the correspondence $\gamma_G : x \mapsto V(x)$ defines the *conformal Gauss map* of the surface $G : \Sigma \rightarrow \mathbb{S}^3$. Since $V = f^\perp$, then γ_G takes values in $G_{3,1}(\mathbb{R}_1^5)$ the Grassmannian of all subspaces of \mathbb{R}_1^5 with signature $(+++-)$. Since V and $V^\perp = \mathbb{R}f$ determine each other then either of them can be used to define the conformal Gauss map of G . Thus for each $x \in \Sigma$, $\gamma_G(x) = \mathbb{R}f(x)$ belongs to the manifold of all spacelike lines through the origin of \mathbb{R}_1^5 which identifies also with $G_{3,1}(\mathbb{R}_1^5)$. Note that the projection $\mathbb{S}_1^4 \rightarrow G_{3,1}(\mathbb{R}_1^5)$ given by $P : p \mapsto \mathbb{R}p$ is a lorentzian double cover. Intersecting the spacelike line $\gamma_G(x) = \mathbb{R}f(x)$ with \mathbb{S}_1^4 we obtain $\{+f(x), -f(x)\} \subset \mathbb{S}_1^4$ which is just the fiber of P over $G(x) \in \mathbb{S}^3$. Thus the surface f and its antipodal $-f$ have the same null Gauss map G . Thus the null Gauss map G can be considered as a pseudo-inverse of the conformal Gauss map γ_G .

4.4. An equation relating κ, s and δ . Let Y be the canonical lift of G respect to a local coordinate z . Then there is a non-zero function τ such that $X = \tau Y$. Using (25), we compute

$$(30) \quad \tau_z Y + \tau Y_z = X_z = e^{-2u}(\xi_2 - \xi_1)f_{\bar{z}} + \sigma X.$$

Hence $\langle X_z, X_{\bar{z}} \rangle = \frac{\tau^2}{2} = \tau^2 \langle Y_z, Y_{\bar{z}} \rangle = e^{-2u}|\xi_2 - \xi_1|^2$, so that

$$(31) \quad \tau = \sqrt{2}e^{-u}|\xi_2 - \xi_1|.$$

Hence we obtain the canonical lift of G in terms of $X = N_1 + N_2$:

$$Y = \frac{e^u}{\sqrt{2}|\xi_2 - \xi_1|}(N_1 + N_2).$$

A routine computation using the structure equations of f shows that Y is in fact independent on any particular choice of a positively oriented lorentzian frame $\{N_1, N_2\}$. On the other hand

$$\begin{aligned} \tau_{zz}Y + 2\tau_z Y_z + \tau Y_{zz} &= X_{zz} = \\ (e^{-2u}(\xi_2 - \xi_1))_z f_{\bar{z}} + e^{-2u}(\xi_2 - \xi_1)(-e^{2u}f + e^{2u}hX) + \\ \sigma_z X + \sigma\{e^{-2u}(\xi_2 - \xi_1)f_{\bar{z}} + \sigma X\}. \end{aligned}$$

Adding and subtracting $\tau \frac{s}{2}Y$ we obtain

$$\begin{aligned} (32) \quad (\tau_{zz} - \tau \frac{s}{2})Y + 2\tau_z Y_z + \tau(Y_{zz} + \frac{s}{2}Y) &= \\ (e^{-2u}(\xi_2 - \xi_1))_z f_{\bar{z}} + e^{-2u}(\xi_2 - \xi_1)(-e^{2u}f + e^{2u}hX) + \\ \sigma_z X + \sigma\{e^{-2u}(\xi_2 - \xi_1)f_{\bar{z}} + \sigma X\}. \end{aligned}$$

Comparing the V^\perp components in this identity we obtain the equality

$$(33) \quad (\xi_1 - \xi_2)f = \tau(Y_{zz} + \frac{s}{2}Y) = \tau\kappa,$$

Inserting the function τ of (31) we obtain a formula for the normal valued Hopf differential of G which makes sense only if the Hopf quadratic deifferential of f is non-zero:

$$(34) \quad \kappa = \frac{(\xi_1 - \xi_2)e^u}{\sqrt{2}|\xi_1 - \xi_2|}f.$$

Using the polar form $(\xi_1 - \xi_2) = |\xi_1 - \xi_2|e^{i\theta}$ the above expression becomes $\kappa = \frac{e^{u+i\theta}}{\sqrt{2}}f$ and so by Remark 3.1 we identify

$$(35) \quad \kappa \equiv \frac{e^{u+i\theta}}{\sqrt{2}}, \quad \text{where } \frac{(\xi_1 - \xi_2)}{|\xi_1 - \xi_2|} = e^{i\theta}.$$

In particular we recover the conformal parameter from κ above:

$$(36) \quad e^{2u} = 2\langle \kappa, \bar{\kappa} \rangle.$$

In [11] it is shown that any section $v \in \Gamma(V \otimes \mathbb{C})$ can be decomposed as follows:

$$(37) \quad v = -\langle v, N \rangle Y - \langle v, Y \rangle N + 2\langle v, Y_{\bar{z}} \rangle Y_z + 2\langle v, Y_z \rangle Y_{\bar{z}}.$$

We use this formula to expand the particular section $f_z \in \Gamma(V \otimes \mathbb{C})$. Since $\tau Y = N_1 + N_2 = X$, it follows $\langle f_z, Y \rangle = 0$. Also from $0 = \langle f, Y_z \rangle_z = \langle f_z, Y_z \rangle + \langle f, Y_{zz} \rangle$, equation (19)-(i), and $\langle f, Y \rangle = 0$, we compute

$$\langle f_z, Y_z \rangle = -\langle f, Y_{zz} \rangle = -\langle f, -\frac{s}{2}Y + \kappa \rangle = -\langle f, \kappa \rangle = -\frac{e^{u+i\theta}}{\sqrt{2}}.$$

On the other hand since $0 = \langle f, Y_{\bar{z}} \rangle_z = \langle f_z, Y_{\bar{z}} \rangle + \langle f, Y_{z\bar{z}} \rangle$, then

$$\langle f_z, Y_{\bar{z}} \rangle = -\langle f, Y_{z\bar{z}} \rangle = |\kappa|^2 \langle Y, f \rangle - \frac{1}{2} \langle N, f \rangle = 0.$$

Also $\langle f, N \rangle = 0$, implies $\langle f_z, N \rangle + \langle f, N_z \rangle = 0$. Hence $\langle f_z, N \rangle = -\langle f, N_z \rangle = -2\langle f, D_{\bar{z}}\kappa \rangle$. Since $D_{\bar{z}}\kappa = (u + i\theta)_{\bar{z}}\kappa$, then

$$\langle f_z, N \rangle = -\sqrt{2}(u + i\theta)_{\bar{z}}e^{u+i\theta}.$$

From these equations and using (37) with $v = f_z$, we obtain

$$(38) \quad f_z = \sqrt{2}e^{u+i\theta}\{(u + i\theta)_{\bar{z}}Y - Y_{\bar{z}}\}.$$

Therefore,

$$(39) \quad f_{z\bar{z}} = \sqrt{2}e^{u+i\theta}\{((u + i\theta)_{\bar{z}})^2 + (u + i\theta)_{\bar{z}\bar{z}} + \frac{\bar{s}}{2}\}Y - \sqrt{2}e^{u+i\theta}\bar{\kappa}.$$

On the other hand using the structure equations of the immersion f and $X = N_1 + N_2 = \tau Y$, we obtain

$$(40) \quad f_{z\bar{z}} = -e^{2u}f + e^{2u}hX = -e^{2u}f + e^{2u}h\tau Y.$$

Note that $\sqrt{2}e^{u+i\theta}\bar{\kappa} = e^{2u}f$, so that equating (39) and (40) gives

$$e^{2u}h\tau = \sqrt{2}e^{u+i\theta}\{((u + i\theta)_{\bar{z}})^2 + (u + i\theta)_{\bar{z}\bar{z}} + \frac{\bar{s}}{2}\}.$$

Inserting the function τ given by (31) in this expression we obtain the following formula:

$$(41) \quad h|\xi_2 - \xi_1|e^{-i\theta} = ((u + i\theta)_{\bar{z}})^2 + (u + i\theta)_{\bar{z}\bar{z}} + \frac{\bar{s}}{2},$$

or conjugating both sides,

$$(42) \quad h(\xi_1 - \xi_2) = ((u - i\theta)_z)^2 + (u - i\theta)_{zz} + \frac{s}{2}.$$

Now recall the connection D on the normal bundle V^\perp . Any section $v \in \Gamma(V^\perp)$ can be written as $v = bf$ for some smooth function b . Thus $d_X(bf) = d_Xbf + bd_Xf$. Condition $df \perp f$ implies $D_Xf = 0$, hence

$$(43) \quad D_X(v) = (d_Xb)f.$$

Thus we may identify $D_X(v) \equiv d_Xb$. Since $\kappa \equiv \frac{e^{u+i\theta}}{\sqrt{2}}$, we compute

$$D_{\bar{z}}D_{\bar{z}}\kappa = \kappa_{\bar{z}\bar{z}} = ((u + i\theta)_{\bar{z}})^2 + (u + i\theta)_{\bar{z}\bar{z}}\kappa.$$

On the other hand since $\overline{h(\xi_1 - \xi_2)}\kappa = \frac{e^u}{\sqrt{2}}h|\xi_2 - \xi_1|$, then $\overline{h(\xi_1 - \xi_2)}\kappa$ is real valued and so equation (41) becomes

$$(44) \quad \kappa_{\bar{z}\bar{z}} + \frac{\bar{s}}{2}\kappa = \operatorname{Re}\left(\overline{h(\xi_1 - \xi_2)}\kappa\right).$$

Equation (44) relates the quadratic differential $h(\xi_1 - \xi_2)dz^2$ of a marginally trapped surface $f : \Sigma \rightarrow \mathbb{S}_1^4$ and the conformal invariants κ, s of its null Gauss map G . Since the quadratic differential $\delta := h(\xi_1 - \xi_2)dz^2$ plays a key role in (44), we call it the δ -differential of the marginally trapped surface f .

Remark 4.3. Equation (44) implies the conformal Codazzi equation $\operatorname{Im}(\kappa_{\bar{z}\bar{z}} + \frac{\bar{s}}{2}\kappa) = 0$. The conformal Gauss equation (20) may be recovered from (42) by a long calculation using Gauss, Codazzi and Ricci's equations (10).

4.5. Congruence. A basic question is to what extent a marginally trapped surface is determined by the conformal invariants of its null Gauss map. We prove,

Theorem 4.3. *Let $f, f' : \Sigma \rightarrow \mathbb{S}_1^4$ be non-isotropic marginally trapped surfaces with null Gauss maps G, G' . If $\kappa = \kappa', s = s'$ then there is an isometry Φ of \mathbb{S}_1^4 such that $\Phi f = f'$. As a consequence of this $\delta = \delta'$.*

Proof. By Theorem 3.1 there is a Moebius transformation $T \in O_+(4, 1)$ of \mathbb{S}^3 such that $TG = G'$. Recall that the Moebius group $O_+(4, 1)$ acts on \mathbb{S}^3 by $T([x]) = [Tx]$, $\forall x \in \mathcal{L}$. Let Y be the canonical lift of G respect to a holomorphic coordinate z , then $Y' = TY$ is the canonical lift of G' respect to z . Since $V = \text{span}\{Y, \text{Re}(Y_z), \text{Im}(Y_z), Y_{zz}\}$, it follows that $TV = V'$ and so $TV^\perp = V'^\perp$. This last equality implies $Tf = \pm f'$ where the sign ambiguity reflects the fact that the sphere congruences determined by f and f' are (modulo Moebius transformations) equal up to orientation. Defining $\Phi = T$, if $Tf = f'$ and $\Phi = -T$, if $Tf = -f'$, then Φ is an isometry of \mathbb{S}_1^4 satisfying $\Phi f = f'$. In particular if T is the identity, then $G = G'$ and so $V^\perp = \mathbb{R}f = \mathbb{R}f'$, which implies $f' = \pm f$.

Let $\{N_1, N_2\} \subset \Gamma(\nu(f))$ be a positively oriented orthonormal frame, then $\{\Phi N_1, \Phi N_2\} \subset \Gamma(\nu(f'))$ is an orthonormal frame. We can choose an orientation on $\nu(f')$ so that $\{\Phi N_1, \Phi N_2\} \subset \Gamma(\nu(f'))$ is a positively oriented normal frame along f' . Since $\vec{H} = h(N_1 + N_2)$ is the mean curvature vector of f , then $\Phi \vec{H} = h(\Phi N_1 + \Phi N_2)$ is the mean curvature vector of f' . Also since $\mathbb{I}(\partial_z, \partial_z) = \xi_1 N_1 + \xi_2 N_2$, then $\mathbb{I}'(\partial_z, \partial_z) = \xi_1 \Phi N_1 + \xi_2 \Phi N_2$ and so $\delta' = h(\xi_1 - \xi_2)dz^2 = \delta$. \square

As a partial converse of the previous Theorem we obtain the following

Lemma 4.4. *Let $f, f' : \Sigma \rightarrow \mathbb{S}_1^4$ be non-isotropic marginally trapped surfaces which induce the same conformal metric. If either*

- i) f, f' are both non-stationary and $\delta = \delta'$, or
 - ii) f, f' are both stationary with $q = q'$,
- then there is an isometry Φ of \mathbb{S}_1^4 such that $\Phi \circ f = f'$.*

Proof: Assume first that f, f' are both non-stationary with $\delta = \delta'$ i.e. $h(\xi_1 - \xi_2)dz^2 = h'(\xi'_1 - \xi'_2)dz^2$, hence $h(\xi_1 - \xi_2) = h'(\xi'_1 - \xi'_2)$. Since h, h' are real and non-zero, we may assume they are both positive (if say $h < 0$, we can replace f by its antipodal $-f$ which has mean curvature function $-h > 0$). Since by hypothesis the Hopf differentials q, q' are never zero, we use the polar form $\xi_1 - \xi_2 = |\xi_1 - \xi_2|e^{i\theta}$ and $\xi'_1 - \xi'_2 = |\xi'_1 - \xi'_2|e^{i\theta'}$. Hence the equality $\delta = \delta'$ implies

$$h|\xi_1 - \xi_2|e^{i\theta} = h'|\xi'_1 - \xi'_2|e^{i\theta'}.$$

It follows that $\theta - \theta' = 2k\pi$ with integer k . Since by hypothesis f and f' induce the same conformal metric, we have $u = u'$ and so (35) implies $\kappa = \kappa'$. On the other hand from $\delta = \delta'$ and (44) it follows that $s = s'$. Thus G, G' have the same conformal invariants κ and s , hence i) follows by applying the preceding Theorem.

If now f, f' are both stationary with $q = q'$, then $|\xi_1 - \xi_2|e^{i\theta} = |\xi'_1 - \xi'_2|e^{i\theta'}$, and so $\theta - \theta'$ is an integer multiple of 2π . Thus since $u = u'$ by hypothesis, (35) implies $\kappa = \kappa'$. Since f, f' are both stationary, then $\delta = \delta' = 0$. Thus from (44), we conclude that $s = s'$, and so G, G' have the same conformal invariants. \square

A conformal immersed surface $\psi : \Sigma \rightarrow \mathbb{S}^3$ is called *constrained Willmore* if it extremizes the Willmore energy functional with respect to variations through conformal immersions [11]. It has been proved in [6] that ψ is constrained Willmore if and only if its conformal invariants κ, s satisfy

$$(45) \quad \kappa_{\bar{z}z} + \frac{\bar{s}}{2}\kappa = \text{Re}(\bar{\eta}\kappa),$$

for some holomorphic quadratic differential ηdz^2 on Σ . Equations (45) and (44) are related. In fact, we have seen before that for an immersed non-isotropic marginally trapped surface $f : \Sigma \rightarrow \mathbb{S}_1^4$ the quantity $\overline{h(\xi_1 - \xi_2)}\kappa$ is real, so that we ask under what conditions is $\delta = h(\xi_1 - \xi_2)dz^2$ holomorphic.

Lemma 4.5. *Let $f : \Sigma \rightarrow \mathbb{S}_1^4$ be a non-isotropic conformally immersed marginally trapped surface with non-zero mean curvature vector. Then the following affirmations are equivalent:*

- i) *The quartic complex differential $Q = \langle f_{zz}, f_{zz} \rangle dz^4$ is holomorphic,*
- ii) *The quadratic complex differential $\delta = h(\xi_1 - \xi_2)dz^2$ is holomorphic,*
- iii) *f has parallel mean curvature vector.*

Proof. Let $\{N_1, N_2\} \subset \Gamma(\nu(f))$ be a positively oriented orthonormal frame, then the quartic differential becomes $Q = (\xi_1^2 - \xi_2^2)dz^4$, where $\xi_1 = \langle f_{zz}, N_1 \rangle$, $\xi_2 = -\langle f_{zz}, N_2 \rangle$ and $\vec{H} = h(N_1 + N_2)$. Since f is marginally trapped Codazzi's equations (10) reduce to

$$e^{-2u}(\xi_{1\bar{z}} + \bar{\sigma}\xi_2) = e^{-2u}(\xi_{2\bar{z}} + \bar{\sigma}\xi_1) = h_z + \sigma h.$$

Using these equations we compute

$$\begin{aligned} (\xi_1^2 - \xi_2^2)_{\bar{z}} &= 2\xi_1\partial_{\bar{z}}\xi_1 - 2\xi_2\partial_{\bar{z}}\xi_2 = \\ &= 2\xi_1(e^{2u}(h_z + \sigma h) - \xi_2\bar{\sigma}) - 2\xi_2(e^{2u}(h_z + \sigma h) - \xi_1\bar{\sigma}) = \\ &= 2e^{2u}(h_z + \sigma h)(\xi_1 - \xi_2). \end{aligned}$$

Since f is non-isotropic q is never zero, so Q is holomorphic if and only if $h_z + \sigma h = 0$, which is just the parallel mean curvature equation (14). This proves i) \Leftrightarrow iii).

Again from Codazzi's equation we get $(\xi_1 - \xi_2)_{\bar{z}} = \bar{\sigma}(\xi_1 - \xi_2)$, which implies $(h(\xi_1 - \xi_2))_{\bar{z}} = (h_{\bar{z}} + \bar{\sigma}h)(\xi_1 - \xi_2)$. Hence $(\xi_1^2 - \xi_2^2)_{\bar{z}} = 2e^{2u}(h(\xi_1 - \xi_2))_{\bar{z}}$, thus δ is holomorphic if and only if Q is holomorphic, and so i) \Leftrightarrow ii). \square

Note for instance that there is no non-isotropic spacelike immersion $f : \mathbb{S}^2 \rightarrow \mathbb{S}_1^4$ with parallel non-zero mean curvature vector. Isotropic marginally trapped surfaces in \mathbb{R}_1^4 and \mathbb{S}_1^4 have been considered in [15].

As a first consequence of equation (44) we deduce that a non-isotropic conformal marginally trapped immersion $f : \Sigma \rightarrow \mathbb{S}_1^4$ has zero mean curvature vector if and only if its null Gauss map $G : \Sigma \rightarrow \mathbb{S}^3$ is a Willmore surface. For, $\vec{H} = 0$ if and only if $\delta \equiv 0$ by (7) if and only if (44) becomes $\kappa_{\bar{z}\bar{z}} + \frac{\bar{s}}{2}\kappa = 0$ which is just the condition for $G : \Sigma \rightarrow \mathbb{S}^3$ being a Willmore surface. We obtain also the following result as consequence of (44):

Theorem 4.6. *Let $f : \Sigma \rightarrow \mathbb{S}_1^4$ be a non-isotropic conformal marginally trapped immersion with null Gauss map G and mean curvature vector $\vec{H} \neq 0$. Then $\nabla^\perp \vec{H} = 0$ if and only if $G : \Sigma \rightarrow \mathbb{S}^3$ is a constrained Willmore surface.*

Proof: The conformal invariants κ, s and the δ -differential of f satisfy equation (44) in which $h(\xi_1 - \xi_2)\kappa$ is real valued. If f has non-zero parallel mean curvature vector then $\delta = h(\xi_1 - \xi_2)dz^2$ is holomorphic by Lemma 4.5. This precisely says that $G : \Sigma \rightarrow \mathbb{S}_1^3$ is constrained Willmore. Conversely if the null Gauss map $G : \Sigma \rightarrow \mathbb{S}_1^3$ of f is a constrained Willmore surface then $\kappa_{\bar{z}\bar{z}} + \frac{\bar{s}}{2}\kappa = \text{Re}(\bar{\eta}\kappa)$, for some holomorphic quadratic differential ηdz^2 . But κ, s uniquely determine the δ -differential of f by Theorem 4.3, so that $\delta = \eta dz^2$. Therefore δ is holomorphic which implies that f has parallel mean curvature vector by Lemma 4.5. \square

5. ONE-PARAMETER DEFORMATIONS AND ASSOCIATED FAMILIES

In classical minimal surface theory an interesting problem is to determine whether a given minimal surface can be deformed in a nontrivial way. The oldest known example is the deformation of the catenoid into the helicoid [29]. We consider here two different non-trivial one-parameter isometric deformations of marginally trapped surfaces in \mathbb{S}_1^4 . Throughout we only consider non-isotropic surfaces.

5.1. The \mathbb{S}^1 -deformation family of marginally trapped surfaces with non-zero parallel mean curvature. In submanifold theory the harmonicity of the Gauss map characterizes submanifolds of spaceforms with parallel mean curvature vector. This is referred to as the Ruh-Vilms property after the well known paper [28]. We obtain here integrable one-parameter deformations of marginally trapped surfaces in \mathbb{S}_1^4 with non-zero parallel mean curvature determined by the spectral symmetry of the harmonic map equation of the ∂ -transform of such surfaces.

Given a conformal immersion $f : \Sigma \rightarrow \mathbb{S}_1^4$ we consider the map

$$\phi : \Sigma \rightarrow \mathbb{CP}^4, \quad x \mapsto [f_z(x)],$$

where $[f_z] \subset \mathbb{C}_1^5$ is the spacelike isotropic complex line generated by f_z . Thus ϕ is well defined since it is independent on the local coordinate z and is called the ∂ -transform of the surface f [31]. Since f is spacelike $[f_z(x)]$ is a spacelike complex line hence it is a point in \mathbb{CP}_1^4 , the open submanifold of \mathbb{CP}^4 consisting of all spacelike complex lines through the origin of \mathbb{C}_1^5 . Moreover since f is conformal ϕ factors through the manifold of isotropic spacelike complex lines in \mathbb{CP}_1^4 which is the complex quadric defined by

$$(46) \quad Q = \{[z] \in \mathbb{CP}_1^4 : z_0^2 + z_1^2 + z_2^2 + z_3^2 - z_4^2 = 0\}.$$

Note that since Q is a complex submanifold, it is totally geodesic in \mathbb{CP}_1^4 . Denote by $L \rightarrow \mathbb{CP}_1^4$ the tautological line bundle whose fiber over a point $l \in \mathbb{CP}_1^4$ is the line l itself and consider the complex line sub bundle $\ell := \phi^*(L) \subset \Sigma \times \mathbb{C}_1^5$. Denote by ℓ^\perp the complementary orthogonal line sub bundle so that $\Sigma \times \mathbb{C}_1^5 = \ell \oplus \ell^\perp$. Any section μ of the trivial bundle $\Sigma \times \mathbb{C}_1^5$, decomposes uniquely as $\mu = \mu_1 + \mu_2$ with $\mu_1 \in \Gamma(\ell)$ and $\mu_2 \in \Gamma(\ell^\perp)$. The projection maps are defined by $\pi_\ell \mu = \mu_1$ and $\pi_{\ell^\perp} \mu = \mu_2$. Since Q is totally geodesic in \mathbb{CP}_1^4 , the map ϕ is harmonic as a map into Q if and only if it is harmonic as a map into \mathbb{CP}_1^4 . Consider on ℓ and ℓ^\perp the Koszul-Malgrange complex structure [16]: a section $s \in \Gamma(\ell)$ (resp. $s \in \Gamma(\ell^\perp)$) is holomorphic if and only if $\pi_\ell(s_{\bar{z}}) = 0$, (resp. $\pi_{\ell^\perp}(s_{\bar{z}}) = 0$). It is known that $\phi : \Sigma \rightarrow \mathbb{CP}_1^4$ is harmonic if and only if the map

$$d\phi(\partial_z) : \ell \rightarrow \ell^\perp, \quad d\phi(\partial_z)\mu = \pi_{\ell^\perp}^\perp(\partial_z\mu)$$

is holomorphic, i.e. it sends holomorphic sections of ℓ , to holomorphic sections of ℓ^\perp [9], [16]. Recall from (5) the structure equations of a conformal immersion $f : \Sigma \rightarrow \mathbb{S}_1^4$ and the corresponding Gauss Codazzi and Ricci's equations (6). The second structure equation $f_{\bar{z}z} = -e^{2u}f + e^{2u}\vec{H}$ implies $\pi_\ell(\partial_{\bar{z}}f_z) = 0$, which says that f_z is a holomorphic section of ℓ . In particular every holomorphic section of ℓ is of the form ζf_z , where ζ a complex holomorphic function. Thus ϕ is harmonic if and only if $\mu := \pi_{\ell^\perp}(f_{zz})$ is a holomorphic section of ℓ^\perp . From the second structure equation it follows that $\mu = \xi_1 N_1 + \xi_2 N_2 = \mathbb{I}(\partial_z, \partial_z) \in \Gamma(\ell^\perp)$. From (5) and Codazzi's equations (6) we compute

$$\begin{aligned} \pi_{\ell^\perp}(\partial_{\bar{z}}\mu) &= \pi_{\ell^\perp}(\partial_{\bar{z}}\xi_1 N_1 + \partial_{\bar{z}}\xi_2 N_2 + \xi_1 \partial_{\bar{z}}N_1 + \xi_2 \partial_{\bar{z}}N_2) = \\ \pi_{\ell^\perp}(\partial_{\bar{z}}\xi_1 N_1 + \partial_{\bar{z}}\xi_2 N_2 + \xi_1(-h_1 f_{\bar{z}} - e^{-2u}\xi_1 f_z + \bar{\sigma} N_2) + \xi_2(h_2 f_{\bar{z}} + e^{-2u}\xi_2 f_z + \bar{\sigma} N_1)) &= \\ e^{-2u}((\partial_{\bar{z}}h_1 + \sigma h_2)N_1 + (\partial_{\bar{z}}h_2 + \sigma h_1)N_2) &= e^{-2u}\nabla_{\partial_{\bar{z}}}^\perp \vec{H}. \end{aligned}$$

Hence the section μ is holomorphic if and only if f has parallel mean curvature vector field and this in turn is equivalent to the harmonicity of ϕ . We summarize the above discussion in the following Lemma which is a manifestation of the characterization due to Ruh-Vilms of submanifolds with parallel mean curvature vector in \mathbb{R}^n and S^n [28]:

Lemma 5.1. *Let $f : \Sigma \rightarrow \mathbb{S}_1^4$ be a conformally immersed surface. Then the Gauss transform $\phi : \Sigma \rightarrow Q \subset \mathbb{CP}_1^4$ of f is harmonic if and only if the surface f has parallel mean curvature vector.*

Remark 5.1. *If f is a non-isotropic marginally trapped with non-zero parallel mean curvature vector then the codimension of f cannot be reduced. This fails to be true if the mean curvature vector of f has non-zero squared norm [13].*

The Lie group $SO_+(4, 1)$ acts transitively by isometries on Q , where we consider a multiple of the Killing metric on Q . Fixing for instance the base point $o := [e_1 - ie_2] \in Q$, then Q

is diffeomorphic to the symmetric quotient $SO_+(4,1)/H$, where H is the stabilizer of the base point $o \in Q$. Consider the involutive isomorphism τ of $SO_+(4,1)$ given by $\tau(F) = EFE$, where $E := \text{diag}(1, -1, -1, 1, 1) \in SO_+(4,1)$. Then the connected component $\text{Fix}(\tau)_0$ of the subgroup of fixed points of τ coincides with H which is isomorphic to $SO(2) \times SO(3,1)$. The (± 1) -eigenspaces of $d\tau_e$ are given respectively by

$$(47) \quad \mathfrak{m} := \left\{ \begin{pmatrix} 0 & a & b & 0 & 0 \\ -a & 0 & 0 & c & d \\ -b & 0 & 0 & e & k \\ 0 & -c & -e & 0 & 0 \\ 0 & d & k & 0 & 0 \end{pmatrix} : a, b, c, d, e, k \in \mathbb{R} \right\}, \quad \mathfrak{h} := \left\{ \begin{pmatrix} 0 & 0 & 0 & m & n \\ 0 & 0 & s & 0 & 0 \\ 0 & -s & 0 & 0 & 0 \\ -m & 0 & 0 & 0 & t \\ n & 0 & 0 & t & 0 \end{pmatrix} : s, t, m, n \in \mathbb{R} \right\},$$

which satisfy $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$, $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$.

We briefly review the loop group formulation of the harmonic map equation for maps into a symmetric space. Let G be a connected semisimple (compact or non-compact) Lie group and assume that G is a matrix group. Let G/H be an inner symmetric space with involution $\tau : G \rightarrow G$ satisfying $(G_\tau)_0 \subseteq H \subseteq G_\tau$, then G/H has a G -invariant non-degenerate symmetric bilinear form [20]. Let $\mathfrak{g} = \text{Lie}(G)$, $\mathfrak{h} = \text{Lie}(H)$ be the Lie algebras of G and H respectively. The involution τ induces a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ into eigenspaces of $d\tau_e$ such that $\mathfrak{h} = \{X : d\tau_e(X) = -X\}$ and $\mathfrak{m} = \{X : d\tau_e(X) = X\}$. It follows that these eigenspaces satisfy $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$.

Let $\psi : \Sigma \rightarrow G/H$ be a smooth map and $F : U \rightarrow G$ a frame of ψ on U , where $U \subset \Sigma$ is a simply connected open subset (if Σ is simply connected then there is always a global frame $F : \Sigma \rightarrow G$). Let $\alpha := F^{-1}dF$ be the Maurer-Cartan one form of F . Then α satisfies the Maurer-Cartan equation $d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$ [10]. By decomposing α into its \mathfrak{h} and \mathfrak{m} parts one obtains

$$\alpha = \alpha_{\mathfrak{h}} + \alpha_{\mathfrak{m}}, \quad \alpha_{\mathfrak{h}} \in \Gamma(\mathfrak{h} \otimes T^*\Sigma), \alpha_{\mathfrak{m}} \in \Gamma(\mathfrak{m} \otimes T^*\Sigma).$$

Also according to the decomposition $T^{\mathbb{C}}\Sigma = T'\Sigma \oplus T''\Sigma$, $\alpha_{\mathfrak{m}}$ decomposes into its $(1,0)$ and $(0,1)$ parts respectively: $\alpha_{\mathfrak{m}} = \alpha'_{\mathfrak{m}} + \alpha''_{\mathfrak{m}}$, hence

$$(48) \quad \alpha = \alpha'_{\mathfrak{m}} + \alpha_{\mathfrak{h}} + \alpha''_{\mathfrak{m}}.$$

It is shown (see [10]) that the harmonic map equation for ψ in terms of α is given by the equation:

$$(49) \quad \bar{\partial}\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{h}} \wedge \alpha'_{\mathfrak{m}}] = 0.$$

There is the following characterization of harmonicity of maps into a symmetric space. Consider the one parameter family of \mathfrak{g} -valued one forms

$$(50) \quad \alpha_{\lambda} := \lambda^{-1}\alpha'_{\mathfrak{m}} + \alpha_{\mathfrak{h}} + \lambda\alpha''_{\mathfrak{m}}, \quad \lambda \in \mathbb{S}^1.$$

Lemma 5.2. [10] $\psi : \Sigma \rightarrow G/H$ is harmonic if and only if

$$(51) \quad d\alpha_{\lambda} + \frac{1}{2}[\alpha_{\lambda} \wedge \alpha_{\lambda}] = 0, \quad \forall \lambda \in \mathbb{S}^1.$$

Let $\phi : \Sigma \rightarrow Q$ be the Gauss transform of an immersed non-isotropic marginally trapped surface $f : \Sigma \rightarrow \mathbb{S}_1^4$ with non-zero parallel mean curvature vector. A frame $F = (F_0, F_1, F_2, N_1, N_2) \in SO_+(4,1)$ (in column notation) is *adapted* to f or *f-adapted* if $Fe_0 = f$ and F_1, F_2 span the tangent space of the immersed surface. Note that if F is *f-adapted*, then N_1, N_2 are normal sections of $\nu(f)$.

The normal bundle $\nu(f)$ is flat since f has parallel mean curvature vector [18], thus we can assume that the normal frame $\{N_1, N_2\}$ in F is positively oriented and ∇^{\perp} -parallel along f . Moreover since f is conformal we can rotate within the tangent plane $\text{span}\{F_1, F_2\}$, if necessary, so that $f_z = \frac{e^u}{\sqrt{2}}(F_1 - iF_2)$. Let $F : \tilde{\Sigma} \rightarrow SO_+(4,1)$ be an *f-adapted* frame, where $\tilde{\Sigma}$ is the universal covering space of Σ . Then the structure equations (9) of the immersed surface f respect to a

coordinate z can be written as $F_z = F.A$, where

$$(52) \quad A = \begin{pmatrix} 0 & -\frac{e^u}{\sqrt{2}} & i\frac{e^u}{\sqrt{2}} & 0 & 0 \\ \frac{e^u}{\sqrt{2}} & 0 & iu_z & -a_1 & a_2 \\ -i\frac{e^u}{\sqrt{2}} & -iu_z & 0 & -ib_1 & ib_2 \\ 0 & a_1 & ib_1 & 0 & \sigma \\ 0 & a_2 & ib_2 & \sigma & 0 \end{pmatrix},$$

where the coefficients in this case are given by

$$(53) \quad \begin{aligned} a_1 &= \frac{e^u h - e^{-u} \xi_1}{\sqrt{2}}, & b_1 &= \frac{-e^u h - e^{-u} \xi_1}{\sqrt{2}}, \\ a_2 &= \frac{e^u h + e^{-u} \xi_2}{\sqrt{2}}, & b_2 &= \frac{-e^u h + e^{-u} \xi_2}{\sqrt{2}}. \end{aligned}$$

Defining $B := \bar{A}$, then $F_{\bar{z}} = FB$ and the compatibility among (9) is just the integrability condition $F_{z\bar{z}} = F_{\bar{z}z}$ which in terms of A, B is given by the matrix differential equation $A_{\bar{z}} - B_z = [A, B]$ encoding Gauss, Codazzi and Ricci's equations (6). In terms of the Maurer-Cartan $\mathfrak{so}(4, 1)$ -valued one form $\alpha := F^{-1}dF = Adz + Bd\bar{z}$ the integrability condition $F_{z\bar{z}} = F_{\bar{z}z}$ is expressed by the Maurer-Cartan equation $d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$, which is just the integrability equation for the existence of an adapted frame solving the equation $F^{-1}dF = \alpha$. Since F is f -adapted then F is also a frame for the Gauss transform ϕ :

$$\phi = [f_z] = \left[\frac{e^u}{\sqrt{2}}(F_1 - iF_2) \right] = [F_1 - iF_2] = [F \cdot (e_1 - ie_2)] = F \cdot [e_1 - ie_2] = F \cdot o$$

We now decompose $A = A_{\mathfrak{m}} + A_{\mathfrak{h}}$, and $B = B_{\mathfrak{m}} + B_{\mathfrak{h}}$, where

$$(54) \quad A_{\mathfrak{m}} = \begin{pmatrix} 0 & -\frac{e^u}{\sqrt{2}} & i\frac{e^u}{\sqrt{2}} & 0 & 0 \\ \frac{e^u}{\sqrt{2}} & 0 & 0 & -a_1 & a_2 \\ -i\frac{e^u}{\sqrt{2}} & 0 & 0 & -ib_1 & ib_2 \\ 0 & a_1 & ib_1 & 0 & 0 \\ 0 & a_2 & ib_2 & 0 & 0 \end{pmatrix}, \quad B_{\mathfrak{m}} = \bar{A}_{\mathfrak{m}},$$

$$(55) \quad A_{\mathfrak{h}} = \text{diag}(0, \begin{pmatrix} 0 & iu_z \\ -iu_z & 0 \end{pmatrix}), \quad B_{\mathfrak{h}} = \bar{A}_{\mathfrak{h}}.$$

Then $A_{\mathfrak{m}}, B_{\mathfrak{m}}$ are $\mathfrak{m}^{\mathbb{C}}$ -valued while $A_{\mathfrak{h}}$ and $B_{\mathfrak{h}}$ are $\mathfrak{h}^{\mathbb{C}}$ -valued. Also since $d\tau_e([A_{\mathfrak{m}}, B_{\mathfrak{m}}]) = [A_{\mathfrak{m}}, B_{\mathfrak{m}}]$, then $[A_{\mathfrak{m}}, B_{\mathfrak{m}}]$ is $\mathfrak{h}^{\mathbb{C}}$ -valued. Note that $\alpha'_{\mathfrak{m}} = A_{\mathfrak{m}}dz$, $\alpha''_{\mathfrak{m}} = B_{\mathfrak{m}}d\bar{z}$ and $\alpha_{\mathfrak{h}} = A_{\mathfrak{h}}dz + B_{\mathfrak{h}}d\bar{z}$ and so the harmonic map equation (49) for the Gauss transform ϕ becomes

$$\partial_{\bar{z}}A_{\mathfrak{m}} + [B_{\mathfrak{h}}, A_{\mathfrak{m}}] = 0.$$

Here the family of one forms α_{λ} (50) is given by

$$(56) \quad \alpha_{\lambda} = \lambda^{-1}A_{\mathfrak{m}}dz + (A_{\mathfrak{h}}dz + B_{\mathfrak{h}}d\bar{z}) + \lambda B_{\mathfrak{m}}d\bar{z}.$$

According to Lemma 5.2 the Gauss transform ϕ is harmonic if and only if α_{λ} satisfies (51). Fixing a point $x_0 \in \tilde{\Sigma}$ and integrating for each $\lambda \in \mathbb{S}^1$

$$(57) \quad dF^{\lambda} = F^{\lambda}\alpha_{\lambda},$$

with initial condition $F^{\lambda}(x_0) = F(x_0) \in H$, one obtains a solution $F^{\lambda} : \tilde{\Sigma} \rightarrow SO_+(4, 1)$, (hence a local solution around any point of Σ) which is called an *extended frame* normalized at x_0 . It is possible to choose the constants of integration so that F^{λ} depends smoothly on $\lambda \in \mathbb{S}^1$ [10]. Since $\alpha_{\lambda=1} = \alpha$, the extended frame satisfies $F^{\lambda=1}(x) = F(x), \forall x \in \tilde{\Sigma}$. In column notation,

$$F^{\lambda} = (F_0^{\lambda}, F_1^{\lambda}, F_2^{\lambda}, N_1^{\lambda}, N_2^{\lambda}).$$

Since the orthonormal frame $\{N_1, N_2\} \subset \Gamma(\nu(f))$ is positively oriented, an elementary argument shows that $\{N_1^{\lambda}, N_2^{\lambda}\}$ is positively oriented $\forall \lambda \in \mathbb{S}^1$.

Now let $f^\lambda := F_0^\lambda = F^\lambda e_0$, i.e. the first column of the extended frame F^λ . Then f^λ is a one parameter deformation of f since at $\lambda = 1$ we recover f : $F^{\lambda=1} e_0 = F \cdot e_0 = f$. We call $f^\lambda, \lambda \in \mathbb{S}^1$ the *associated family* of the marginally trapped surface f . Observe that

$$(58) \quad f_z^\lambda := F_z^\lambda e_0 = F^\lambda (\lambda^{-1} A_m + A_h) e_0 = \lambda^{-1} \frac{e^u}{\sqrt{2}} F^\lambda (e_1 - ie_2),$$

hence F^λ is adapted to f^λ . From (58) we compute

$$(59) \quad \begin{aligned} \langle f_z^\lambda, f_z^\lambda \rangle &= \langle \lambda^{-1} \frac{e^u}{\sqrt{2}} (e_1 - ie_2), \lambda^{-1} \frac{e^u}{\sqrt{2}} (e_1 - ie_2) \rangle = 0. \\ \langle f_z^\lambda, f_{\bar{z}}^\lambda \rangle &= \langle F^\lambda (\lambda^{-1} A_m + A_h) e_0, F^\lambda (\lambda B_m + B_h) e_0 \rangle = \\ &= \langle \lambda^{-1} \frac{e^u}{\sqrt{2}} (e_1 - ie_2), \lambda \frac{e^u}{\sqrt{2}} (e_1 + ie_2) \rangle = e^{2u}. \end{aligned}$$

Hence f^λ is a conformal spacelike immersion inducing the same conformal metric for any $\lambda \in \mathbb{S}^1$.

Let $\phi^\lambda := F^\lambda \cdot o$. Since $\phi^{\{\lambda=1\}} = \phi$, ϕ^λ is a one parameter deformation of ϕ . The family of maps ϕ^λ is called the *associated family* of the harmonic Gauss transform ϕ [10]. Note that from (58) it follows that ϕ^λ is the Gauss transform of f^λ :

$$\phi^\lambda = F^\lambda \cdot o = [F^\lambda (e_1 - ie_2)] = [\lambda f_z^\lambda] = [f_z^\lambda].$$

Hence ϕ^λ takes values in the complex quadric Q . Moreover since $(\alpha_\lambda)'_m = \lambda^{-1} \alpha'_m$, $(\alpha_\lambda)''_m = \lambda \alpha''_m$, and $(\alpha_\lambda)_h = \alpha_h$, it follows that α_λ satisfies equation (49), thus each $\phi^\lambda : \Sigma \rightarrow Q$ is harmonic hence each member f^λ has parallel mean curvature.

We claim that f^λ is marginally trapped for any $\lambda \in \mathbb{S}^1$. Denote by \vec{H}_λ the mean curvature vector of f^λ . Since f^λ is conformal and spacelike, it follows that

$$(60) \quad f_{z\bar{z}}^\lambda = -e^{2u} f^\lambda + e^{2u} \vec{H}_\lambda,$$

hence from (8) we obtain

$$(61) \quad \vec{H}_\lambda = e^{-2u} \langle f_{z\bar{z}}^\lambda, N_1^\lambda \rangle N_1^\lambda - e^{-2u} \langle f_{z\bar{z}}^\lambda, N_2^\lambda \rangle N_2^\lambda.$$

On the other hand the structure equations of f^λ are expressed by the matrix equation $F_z^\lambda = F^\lambda (\lambda^{-1} A_m + A_h)$, which is equivalent to the system

$$(62) \quad \begin{aligned} f_z^\lambda &= \frac{1}{\lambda} \frac{e^u}{\sqrt{2}} F_1^\lambda - i \frac{1}{\lambda} \frac{e^u}{\sqrt{2}} F_2^\lambda, \\ \partial_z F_1^\lambda &= -\frac{1}{\lambda} \frac{e^u}{\sqrt{2}} f^\lambda - i u_z F_2^\lambda + \frac{1}{\lambda} a_1 N_1^\lambda + \frac{1}{\lambda} a_2 N_2^\lambda, \\ \partial_z F_2^\lambda &= i \frac{1}{\lambda} \frac{e^u}{\sqrt{2}} f^\lambda + i u_z F_1^\lambda + i \frac{1}{\lambda} b_1 N_1^\lambda + i \frac{1}{\lambda} b_2 N_2^\lambda, \\ \partial_z N_1^\lambda &= -\frac{1}{\lambda} a_1 F_1^\lambda - i \frac{1}{\lambda} b_1 F_2^\lambda + \sigma N_1^\lambda, \\ \partial_z N_2^\lambda &= \frac{1}{\lambda} a_2 F_1^\lambda + i \frac{1}{\lambda} b_2 F_2^\lambda + \sigma N_2^\lambda, \end{aligned}$$

from which it follows

$$\begin{aligned} \langle f_{z\bar{z}}^\lambda, N_1^\lambda \rangle &= -\langle f_{\bar{z}}^\lambda, \partial_z N_1^\lambda \rangle = (a_1 - b_1) \frac{e^u}{\sqrt{2}} = e^{2u} h, \\ \langle f_{z\bar{z}}^\lambda, N_2^\lambda \rangle &= -\langle f_{\bar{z}}^\lambda, \partial_z N_2^\lambda \rangle = (b_2 - a_2) \frac{e^u}{\sqrt{2}} = -e^{2u} h. \end{aligned}$$

From (61) we obtain $\vec{H}_\lambda = h(N_1^\lambda + N_2^\lambda)$, which shows that f^λ is marginally trapped for every $\lambda \in \mathbb{S}^1$, with $h^\lambda = h$.

On the other hand since $\xi_1^\lambda = \langle f_{z\bar{z}}^\lambda, N_1^\lambda \rangle$ and $\xi_2^\lambda = -\langle f_{z\bar{z}}^\lambda, N_2^\lambda \rangle$, then from (62) we obtain

$$(63) \quad \begin{aligned} \xi_1^\lambda &= \langle f_{z\bar{z}}^\lambda, N_1^\lambda \rangle = -\langle f_z^\lambda, \partial_z N_1^\lambda \rangle = \lambda^{-2} \frac{e^u}{\sqrt{2}} (a_1 + b_1) = \lambda^{-2} \xi_1, \\ \xi_2^\lambda &= -\langle f_{z\bar{z}}^\lambda, N_2^\lambda \rangle = \langle f_z^\lambda, \partial_z N_2^\lambda \rangle = \lambda^{-2} \frac{e^u}{\sqrt{2}} (a_2 + b_2) = \lambda^{-2} \xi_2. \end{aligned}$$

Hence the $(2, 0)$ part of the second fundamental form of f_λ is given by,

$$(64) \quad II^\lambda(\partial_z, \partial_z) = \lambda^{-2} \xi_1 N_1^\lambda + \lambda^{-2} \xi_2 N_2^\lambda.$$

From the above expression we see that $Q_\lambda = \lambda^{-2} Q$, where $Q_\lambda = \langle f_{z\bar{z}}^\lambda, f_{z\bar{z}}^\lambda \rangle dz^4$. Hence f^λ is non-isotropic for every $\lambda \in \mathbb{S}^1$. We collect these facts in the following,

Proposition 5.1. *Let $f : \Sigma \rightarrow \mathbb{S}_1^4$ be a non-isotropic conformal marginally trapped immersion with non-zero parallel mean curvature vector and let f^λ its associated family obtained above which is defined on a simply connected open neighborhood of each point of Σ . Then each member f^λ is a conformal immersion inducing the same conformal metric for any $\lambda \in \mathbb{S}^1$. Moreover, f^λ is a non-isotropic marginally trapped surface with non-zero parallel mean curvature vector.*

Since $f : \Sigma \rightarrow \mathbb{S}_1^4$ has non-zero parallel mean curvature vector its normal bundle is flat thus by Lemma 4.2 the Hopf quadratic differential $q = (\xi_1 - \xi_2)dz^2$ is holomorphic. Since f is non-isotropic q is never zero on Σ , thus for any point $x \in \Sigma$ there is a local coordinate z such that $q = cdz^2$, for a non-zero real constant c . By (34) κ is real in the same coordinate z and so the null Gauss map $G : \Sigma \rightarrow \mathbb{S}^3$ of f is isothermic.

The conformal invariants $\kappa_\lambda, s_\lambda$ and δ -differential of the associated family f^λ can be computed as follows. From (63) and (64) we first obtain the Hopf differential of f^λ :

$$(65) \quad q_\lambda = \lambda^{-2}cdz^2 = \lambda^{-2}q.$$

Since $h_\lambda = h$, from the above expression we obtain

$$(66) \quad \delta_\lambda = hq_\lambda = \lambda^{-2}chdz^2 = \lambda^{-2}\delta.$$

In polar form the Hopf differential q_λ is given by, $q_\lambda = |c|e^{i\theta(\lambda)}dz^2 = \lambda^{-2}|c|e^{i\theta}dz^2$. Thus $e^{i\theta(\lambda)} = \lambda^{-2}e^{i\theta}$ so that if $\lambda = e^{i\varphi}$ then

$$(67) \quad \theta(\lambda) = \theta - 2\varphi.$$

Since λ does not depend on z , neither does φ and so $\theta(\lambda)_z = \theta_z$, and $\theta(\lambda)_{zz} = \theta_{zz}$. Taking this into account and applying (42) to the conformal invariants $\kappa_\lambda, s_\lambda$ and the delta differential δ_λ of f^λ , we obtain,

$$(68) \quad ch\lambda^{-2} = ((u - i\theta)_z)^2 + (u - i\theta)_{zz} + \frac{s_\lambda}{2}.$$

Combining the above equation with (42) gives the Schwartzian derivative of G_λ :

$$(69) \quad s_\lambda = s + 2(\lambda^{-2} - 1)ch.$$

Also from (35) κ_λ identifies with $\frac{e^{u+i\theta(\lambda)}}{\sqrt{2}}$, thus from (67) we obtain

$$(70) \quad \kappa_\lambda = \frac{e^{u+i(\theta-2\varphi)}}{\sqrt{2}} = \lambda^{-2}\kappa.$$

A straightforward computation using (68), (69) and (70) shows that $\kappa_\lambda, s_\lambda, \delta_\lambda$ obey the fundamental equation (44) namely,

$$(\kappa_\lambda)_{\bar{z}z} + \frac{\bar{s}_\lambda}{2}\kappa_\lambda = ch\bar{\lambda}^{-2}\kappa_\lambda, \quad \forall \lambda \in \mathbb{S}^1,$$

in which $ch\bar{\lambda}^{-2}\kappa_\lambda = ch\kappa$, hence it is real valued for every $\lambda \in \mathbb{S}^1$. In particular $\kappa_\lambda, s_\lambda$ obey the conformal Codazzi equation:

$$\text{Im} \left((\kappa_\lambda)_{\bar{z}z} + \frac{\bar{s}_\lambda}{2}\kappa_\lambda \right) = 0, \quad \forall \lambda \in \mathbb{S}^1.$$

Since f has parallel mean curvature vector, δ is holomorphic hence from (68), (69), (70) it easily follows that $\kappa_\lambda, s_\lambda$ obey the conformal Gauss equation:

$$\frac{(s_\lambda)_{\bar{z}}}{2} = 3(\bar{\kappa}_\lambda)_z \cdot \kappa_\lambda + \bar{\kappa}_\lambda(\kappa_\lambda)_z.$$

Since λ does not depend on z and δ is holomorphic, then $\delta_\lambda = \lambda^{-2}\delta$ is holomorphic for any $\lambda \in \mathbb{S}^1$.

We have proved the following

Proposition 5.2. *Let f^λ be the associated family of a non-isotropic marginally trapped surface $f : \Sigma \rightarrow \mathbb{S}_1^4$ with non-zero parallel mean curvature vector. Then for any $\lambda \in \mathbb{S}^1$ the conformal invariants and δ -differential of f^λ are given by*

$$(71) \quad \kappa_\lambda = \lambda^{-2} \kappa, \quad s_\lambda = s + 2(\lambda^{-2} - 1)ch, \quad \delta_\lambda = \lambda^{-2} \delta,$$

where $q = cdz^2$ and $\delta = chdz^2$.

Moreover, the system consisting of (44) and the conformal Gauss and Codazzi equations (20) is invariant under the spectral symmetry determined by (71).

Note that as consequence of (71) the members of the associated family f^λ are non-congruent, hence the deformation $f \mapsto f^\lambda$ is non-trivial. It also follows that the isothermic condition is preserved by the spectral symmetry (71): if κ is real for some coordinate z then in the new coordinate $w = \frac{1}{\lambda}z$ κ_λ is real since $\kappa_\lambda dz^2 = \kappa dw^2$.

Remark 5.2. In [11] the authors obtain the following slightly different symmetry for the conformal Gauss and Codazzi equations of a constrained Willmore surface $\psi : \Sigma \rightarrow \mathbb{S}^3$:

$$(72) \quad \kappa_\lambda = \lambda \kappa, \quad s_\lambda = s + (\lambda^2 - 1)\eta, \quad \eta_\lambda = \lambda^2 \eta,$$

where ηdz^2 is an holomorphic quadratic differential satisfying $\kappa_{\bar{z}z} + \frac{\bar{s}}{2}\kappa = \text{Re}(\bar{\eta}\kappa)$.

5.2. The Calapso-Bianchi associated family of marginally trapped surfaces with flat normal bundle. We construct an integrable deformation of non-isotropic marginally trapped surfaces with flat normal bundle which is related to the so-called Calapso-Bianchi T-transform of isothermic surfaces in \mathbb{S}^3 [11]. The class of marginally trapped surfaces with flat normal bundle in \mathbb{S}_1^4 includes those with non-zero parallel mean curvature vector and also the spacelike isothermic surfaces introduced by P. Wang in [30].

Recall that a conformally immersed surface $\psi : \Sigma \rightarrow \mathbb{S}^3$ is *isothermic* if away from umbilics, it can be conformally parameterized by its curvature lines. In terms of its conformal invariants a surface ψ is isothermic if each point in Σ has a coordinate z for which κ is real: $\kappa = \bar{\kappa}$ [11], [32]. In this case the conformal Gauss and Codazzi's equations (20) away of umbilic points reduce to

$$(73) \quad \begin{aligned} s_{\bar{z}} &= 4(\kappa^2)_z, \\ \text{Im}(\kappa_{\bar{z}z} + \frac{1}{2}\bar{s}\kappa) &= 0. \end{aligned}$$

Thus away from umbilic points κ is non-zero and so both equations combine into Calapso's equation: $\Delta(\frac{\kappa_{xy}}{\kappa}) + 8(\kappa^2)_{xy} = 0$. The Calapso-Bianchi T-transform acts on an isothermic surface $\psi : \Sigma \rightarrow \mathbb{S}^3$ by deforming the schwartzian s and keeping κ unchanged:

$$(74) \quad s_t = s + t, \quad \kappa_t = \kappa, \quad t \in \mathbb{R},$$

thus giving rise to the so-called associated family ψ_t [11].

Let $f : \Sigma \rightarrow \mathbb{S}_1^4$ be a non-isotropic marginally trapped surface with flat normal bundle. Then by Lemma 4.2 for every point $x \in \Sigma$ there is a local coordinate z such that $q = cdz^2$ for a non-zero real constant c . Thus κ is real in the same coordinate and so the null Gauss map $G : \Sigma \rightarrow \mathbb{S}^3$ of f is isothermic. Conversely, if G is isothermic, then κ and so q is real in some coordinate z . Hence f has flat normal bundle if and only if q is constant i.e. $q = cdz^2$ for some non-zero real constant c .

The structure equations of f read (9) in which $\xi_1 = \xi + c$, $\xi_2 = \xi$, $\sigma = 0$, where the positively oriented orthonormal frame $\{N_1, N_2\}$ is ∇^\perp -parallel. The compatibility equations (10) reduce in this case to

$$(75) \quad \begin{aligned} 2u_{\bar{z}z} &= -e^{2u} + e^{-2u}(2c\text{Re}(\xi) + c^2), \\ \xi_{\bar{z}} &= e^{2u}h_z, \\ 0 &= \text{Im}((\xi + c)\xi), \end{aligned}$$

where $2\xi c + c^2 \neq 0$ since f is non-isotropic. If h is a non-zero constant, then f has non-zero parallel mean curvature vector field and its null Gauss map $G : \Sigma \rightarrow \mathbb{S}^3$ is isothermic and constrained Willmore. On the other hand if h is a non-constant function satisfying (75), then f has flat normal bundle and non-parallel mean curvature vector field.

Since our considerations are local we consider an f -adapted frame $F = (F_0, F_1, F_2, N_1, N_2) \in SO_+(4, 1)$ defined on the universal covering space $\tilde{\Sigma}$. Then the structure equations of f read $F_z = FA$, where the coefficients of the matrix A in (52) are given in this case by

$$\begin{aligned} a_1 &= \frac{e^{-u}(\xi+c)+e^u h}{\sqrt{2}}, & b_1 &= \frac{e^{-u}(\xi+c)-e^u h}{\sqrt{2}}, & \sigma &= 0, \\ a_2 &= \frac{e^{-u}\xi+e^u h}{\sqrt{2}}, & b_2 &= \frac{-e^{-u}\xi-e^u h}{\sqrt{2}}. \end{aligned}$$

We now introduce a one-parameter family of matrices given by

$$(76) \quad A^t = \begin{pmatrix} 0 & -\frac{e^u}{\sqrt{2}} & i\frac{e^u}{\sqrt{2}} & 0 & 0 \\ \frac{e^u}{\sqrt{2}} & 0 & iu_z & -a_1^t & a_2^t \\ -i\frac{e^u}{\sqrt{2}} & -iu_z & 0 & -ib_1^t & ib_2^t \\ 0 & a_1^t & ib_1^t & 0 & 0 \\ 0 & a_2^t & ib_2^t & 0 & 0 \end{pmatrix}, \quad B^t = \overline{A^t} \in \mathfrak{so}(4, 1)^\mathbb{C}, \quad t \in \mathbb{R},$$

with coefficients

$$(77) \quad \begin{aligned} a_1^t &= \frac{e^{-u}(\xi+c)+e^u h^t}{\sqrt{2}}, & b_1^t &= \frac{e^{-u}(\xi+c)-e^u h^t}{\sqrt{2}}, \\ a_2^t &= \frac{e^{-u}\xi+e^u h^t}{\sqrt{2}}, & b_2^t &= \frac{-e^{-u}\xi-e^u h^t}{\sqrt{2}}, \end{aligned}$$

where

$$(78) \quad h^t := h + \frac{t}{2c}, \quad c \in \mathbb{R}^\times, \quad t \in \mathbb{R}.$$

Note that for $t = 0$ we recover A , i.e. $A^{t=0} = A$.

Lemma 5.3. *Define a one parameter family of $\mathfrak{so}(4, 1)$ -valued one-forms by*

$$(79) \quad \alpha_t := A^t dz + B^t d\bar{z}, \quad t \in \mathbb{R}.$$

Then α_t coincides with α for $t = 0$ and it satisfies the Maurer-Cartan equation

$$(80) \quad d\alpha_t + \frac{1}{2}[\alpha_t \wedge \alpha_t] = 0, \quad \forall t \in \mathbb{R},$$

if and only if u, ξ, h satisfy (75).

Proof. Since $B^t = \overline{A^t}$ then α_t is $\mathfrak{so}(4, 1)$ -valued for every $t \in \mathbb{R}$. On the other hand $d\alpha_t + \frac{1}{2}[\alpha_t \wedge \alpha_t] = 0$ is equivalent to $(A^t)_z - (B^t)_z = [A^t, B^t]$ which in turn is equivalent to

$$\begin{aligned} 2u_{z\bar{z}} &= -e^{2u} + e^{-2u}(2c\operatorname{Re}(\xi) + c^2), \\ \xi_{\bar{z}} &= e^{2u}(h^t)_z, \\ 0 &= \operatorname{Im}((\xi + c)\xi). \end{aligned}$$

Since $(h^t)_z = h_z$ for any $t \in \mathbb{R}$, the above system is invariant under the symmetry (78) and it is equivalent to (75). \square

Since we work locally, we may transfer the situation to the universal covering space $\tilde{\Sigma}$ of Σ (note that the case $\tilde{\Sigma} = \mathbb{S}^2$ is excluded, otherwise being q holomorphic it would vanish). Thus we can integrate the Maurer-Cartan equation (80) on $\tilde{\Sigma}$ for each t , obtaining a solution $F^t : \tilde{\Sigma} \rightarrow SO_+(4, 1)$, which is unique up to left translation by a constant element in $SO_+(4, 1)$. Thus F^t satisfies

$$(81) \quad (F^t)^{-1} dF^t = \alpha_t, \quad F^0 = F,$$

since $\alpha_0 = \alpha$. According to [10], [19] it is possible to choose the constants of integration so that $t \mapsto F^t(x)$ is C^∞ for every $x \in \tilde{\Sigma}$. Denote by $F^t := (F_0^t, F_1^t, F_2^t, N_1^t, N_2^t)$ in column notation. Since

$N_2^0 = N_2$ is future pointing, then by continuity N_2^t is future pointing for every t . Moreover, since $\{N_1^0, N_2^0\} = \{N_1, N_2\}$ is positively oriented, then an elementary continuity argument shows that $\{N_1^t, N_2^t\}$ is positively oriented for every $t \in \mathbb{R}$.

Define $f^t := F^t.e_0$, the first column of F^t , then

$$(82) \quad f_z^t = F_z^t e_0 = F^t A^t.(e_1 - ie_2) = \frac{e^u}{\sqrt{2}} F^t(e_1 - ie_2),$$

from which we compute

$$\begin{aligned} \langle f_z^t, f_{\bar{z}}^t \rangle &= \frac{e^{2u}}{2} \langle F^t(e_1 - ie_2), F^t(e_1 + ie_2) \rangle = e^{2u}, \\ \langle f_z^t, f_z^t \rangle &= \frac{e^{2u}}{2} \langle F^t(e_1 - ie_2), F^t(e_1 - ie_2) \rangle = 0, \end{aligned}$$

hence f^t is a conformal spacelike immersion which induces the same (conformal) metric for any t . Since $f^{t=0} = f$, f^t is a one parameter deformation of f . Also from (81) and (82) we obtain

$$f_{z\bar{z}}^t = u_z f_z^t + \frac{e^u}{\sqrt{2}} F^t B(e_1 - ie_2), \quad f_{zz}^t = u_z f_z^t + \frac{e^u}{\sqrt{2}} F^t A(e_1 - ie_2),$$

which, from the structure of the matrices A^t, B^t , become,

$$(83) \quad f_{z\bar{z}}^t = -e^{2u} f^t + e^{2u} h^t (N_1^t + N_2^t), \quad f_{zz}^t = 2u_z f_z^t + (\xi + c) N_1^t + \xi N_2^t.$$

Hence the mean curvature vector of f^t is given by $\vec{H}_t = h^t (N_1^t + N_2^t)$ and so f^t is marginally trapped. Also from (83) we see that

$$\langle f_{zz}^t, f_{zz}^t \rangle = (\xi + c)^2 - \xi^2 = 2\xi c + c^2 = \langle f_{zz}, f_{zz} \rangle. \quad \forall t \in \mathbb{R},$$

hence f^t is non-isotropic. On the other hand F^t is adapted to f^t since $F_z^t = F^t A^t$. From this equation we extract

$$\partial_z N_1^t = -a_1^t F_1^t - ib_1^t F_2^t, \quad \partial_z N_2^t = a_2^t F_1^t + ib_2^t F_2^t,$$

which shows that f^t has flat normal bundle for every t and that $\{N_1^t, N_2^t\}$ is a parallel orthonormal frame with respect to the normal connection ∇_t^\perp of $\nu(f^t)$.

Equation (44) relating the conformal invariants and the δ differential of f reads

$$(84) \quad \kappa_{\bar{z}\bar{z}} + \frac{\bar{s}}{2} \kappa = ch\kappa, \quad c \in \mathbb{R}^\times, \quad \delta = chdz^2.$$

The deformation family f^t obtained above is locally defined on Σ and is related to (74) hence we call $f \mapsto f^t$ the *Calapso-Bianchi transformation* of the marginally trapped surface f .

Since $(s_t)_z = s_{\bar{z}}$, then by Theorem 3.1 κ, s_t determine a unique (up to Moebius transformations of the sphere) conformal immersed isothermic surface $G^t : \Sigma \rightarrow \mathbb{S}^3$. Since for $t = 0$ we recover s in (74), G^t is the associated family of G or the T-transform of the isothermic surface $G : \Sigma \rightarrow \mathbb{S}^3$. We claim that G^t is the null Gauss map of f^t . In fact, from (83) it follows that $q = cdz^2$ is the Hopf differential of f^t . Since f^t induce the same conformal metric for all t , then θ in formula (35) must be an integer multiple of 2π , and so $\kappa = \frac{e^u}{\sqrt{2}}$ is the (common) normal Hopf differential of the null Gauss map of all f^t . Inserting (74) into (84) yields,

$$(85) \quad \kappa_{\bar{z}\bar{z}} + \frac{\bar{s}_t}{2} \kappa = c(h + \frac{t}{2c}) = ch^t \kappa, \quad \delta_t = ch^t dz^2,$$

where $\delta_t = ch^t dz^2$ is just the delta differential of f^t . Thus the above equation is the evolution of (84) and so κ, s_t are the conformal invariants of the null Gauss map of f^t . Thus G^t has conformal invariants κ, s_t and so it coincides up to a Moebius transformation of \mathbb{S}^3 with the null Gauss map of f^t which is isothermic since κ is real.

The transformation $f \mapsto f^t$ also preserves marginally trapped surfaces which are isothermic or have parallel second fundamental form. For instance if f is isothermic then for each $x \in \Sigma$ there is a local coordinate z for which $\Omega = \xi_1 N_1 dz^2 + \xi_2 N_2 dz^2$ is real valued, that is ξ_1, ξ_2 are real valued. Thus by Ricci's equation f has flat normal bundle and so the Hopf differential q is holomorphic by Lemma 4.2 and so $q = cdz^2$ for a non-zero real constant c , with $\xi_1 - \xi_2 = c$. Hence the function

ξ in (75) satisfying $\xi_1 = \xi + c, \xi_2 = \xi$ must be also real valued. Thus from (77) it follows that the normal vector Hopf differential Ω^t of f^t is also real valued in the same coordinate z , which shows that f^t is isothermic for any $t \in \mathbb{R}$.

On the other hand if f has non-zero parallel mean curvature vector then it has flat normal bundle by [18]. Thus there is a local positive ∇^\perp -parallel orthonormal frame $\{N_1, N_2\} \subset \Gamma(\nu(f))$ such that $0 = \nabla_{\partial_z}^\perp \vec{H} = h_z(N_1 + N_2)$, thus h is constant. Since h_t is defined by (78) it satisfies $(h^t)_z = h_z$, then $(h^t)_z = 0$ for all $t \in \mathbb{R}$ which shows that f^t has (non-zero) parallel mean curvature vector for any $t \in \mathbb{R}$. We summarize our discussion in the following

Theorem 5.4. *Let $f : \Sigma \rightarrow \mathbb{S}_1^4$ be a non-isotropic conformal marginally trapped immersion with flat normal bundle. Let $f^t : \Sigma \rightarrow \mathbb{S}_1^4$ be the Calapso-Bianchi deformation family of f obtained by integration of (81). Then on Σ each f^t is locally defined conformal non-isotropic marginally trapped immersion with flat normal bundle whose null Gauss map G^t is isothermic for any $t \in \mathbb{R}$. Moreover, the transformation $f \mapsto f^t$ preserves isothermic surfaces and surfaces with non-zero parallel mean curvature vector.*

5.3. An extended deformation. Non-isotropic marginally trapped conformal immersed surfaces in \mathbb{S}_1^4 with non-zero parallel mean curvature vector have flat normal bundle [18] and have isothermic and constrained Willmore null Gauss maps into \mathbb{S}^3 by Theorem 4.6. In the previous section we considered two different one parameter deformations for such surfaces, namely $f^\lambda, \lambda \in \mathbb{S}^1$ and $f^t, t \in \mathbb{R}$. Motivated by [11] we show that it is possible to unify both deformations by defining an (extended) action of $\mathbb{C} - \{0\}$ on the set of non-isotropic marginally trapped surfaces with non-zero parallel mean curvature vector.

Let $f : \Sigma \rightarrow \mathbb{S}_1^4$ be a non-isotropic conformally immersed marginally trapped surface with non-zero parallel mean curvature vector and κ, s be the conformal invariants of f , δ -differential $\delta = chdz^2$, with $h = \text{const} \neq 0$ and quadratic Hopf differential $q = cdz^2$, for real constant $c \neq 0$. We extend the symmetry (71) for $\lambda \in \mathbb{C} - \{0\}$ by defining

$$(86) \quad \kappa_\lambda = |\lambda|^2 \lambda^{-2} \kappa, \quad s_\lambda = s + 2(\lambda^{-2} - 1)ch, \quad \delta_\lambda = \lambda^{-2} \delta.$$

Thus for $|\lambda| = 1$ above we recover (71). Moreover, since $ch\kappa$ is real, a straightforward calculation shows that $\kappa_\lambda, s_\lambda, \delta_\lambda$ above satisfy (44) and the conformal Gauss and Codazzi's equation (20) for every $\lambda \in \mathbb{C} - \{0\}$. Thus $\kappa_\lambda, s_\lambda, \delta_\lambda$ determine the extended associated family f^λ which for $|\lambda| = 1$ restricts to the associated family obtained in the previous section.

We describe the deformation (86) of a non-isotropic marginally trapped torus in \mathbb{S}_1^4 with non-zero parallel mean curvature. The image of the null Gauss map in this case is an isothermic constrained Willmore torus in \mathbb{S}^3 and by a result of Richter [27] (see also [11]) it can be immersed as a surface of constant mean curvature in some riemannian space form.

Let $f : \Sigma \rightarrow \mathbb{S}_1^4$ be a conformal non-isotropic marginally trapped immersion with non-zero parallel mean curvature then $Q = \langle f_{zz}, f_{zz} \rangle dz^4$ is holomorphic and non-zero. Away from the isolated zeros of Q it is possible to choose a local coordinate z such that $Q = dz^4$, or $\langle f_{zz}, f_{zz} \rangle = 1$. When $\Sigma = T^2$ is a 2-torus then Q has no zeros at all (otherwise Q would be identically zero). Thus $Q = dz^4$, where z is a global coordinate on the universal covering \mathbb{C} of T^2 which determines a bi-holomorphism $T^2 \cong \mathbb{C}/\Gamma$ for some lattice $\Gamma_0 \subset \mathbb{C}$. We choose a positively oriented orthonormal lorentzian frame $\{N_1, N_2\} \subset \Gamma(\nu(f))$ such that

$$f_{zz} = 2u_z f_z + \cosh(C)N_1 + \sinh(C)N_2,$$

where $C = \rho + i\Theta$ is a complex function. The new positively oriented lorentzian frame $\{N'_1, N'_2\}$ given by

$$\begin{aligned} N'_1 &= \cosh(\rho)N_1 + \sinh(\rho)N_2, \\ N'_2 &= \sinh(\rho)N_1 + \cosh(\rho)N_2, \end{aligned}$$

has structure function $\sigma' = 0$ and so $\{N'_1, N'_2\}$ is ∇^\perp -parallel along f . Also since

$$f_{zz} = 2u_z f_z + \cos(\Theta)N'_1 + i \sin(\Theta)N'_2,$$

then Ricci's equation now becomes $0 = \cos(\Theta) \sin(\Theta)$, of which $\Theta = 0$ is a solution. For simplicity we drop the primes and keep denoting by $\{N_1, N_2\}$ this new ∇^\perp -parallel normal frame. The structure equations of f become

$$(87) \quad \begin{aligned} f_{zz} &= 2u_z f_z + N_1, \\ f_{\bar{z}\bar{z}} &= -e^{2u} f + e^{2u} h(N_1 + N_2), \\ \partial_z N_1 &= -h f_z - e^{-2u} f_{\bar{z}}, \\ \partial_z N_2 &= h f_z, 0 \neq h = \text{const.} \end{aligned}$$

with compatibility given by the Sinh-Gordon equation $2u_{\bar{z}\bar{z}} = -e^{2u} + e^{-2u}$, of which $u : \mathbb{C} \rightarrow \mathbb{R}$ is a doubly periodic solution with respect to the lattice $\Gamma_0 \subset \mathbb{C}$. Solutions to the Sinh-Gordon equation are obtained by applying the finite-gap integration method from theta functions defined on auxiliary hyperelliptic Riemann surfaces which arise from inverse scattering theory [5].

Since the mean curvature vector is lightlike and non-zero the codimension of the surface f cannot be reduced. Moreover $(N_1 + N_2)_z = -e^{-2u} f_{\bar{z}}$ implies that f cannot lie in any singular hypersurface of \mathbb{S}_1^4 . From (87) the Hopf differential of f is given by $q = dz^2$, hence θ must be an integer multiple of 2π in (35) and so $\kappa = \frac{e^u}{\sqrt{2}}$ which says that $G : T^2 \rightarrow \mathbb{S}^3$ is an isothermic and constrained Willmore surface since h is a non-zero constant. The fundamental equation (44) becomes $\kappa_{\bar{z}\bar{z}} + \frac{\bar{s}}{2}\kappa = h\kappa$, $\delta = h dz^2$.

We see from (86) that $\kappa_\lambda = |\lambda|^2 \lambda^{-2} \frac{e^u}{\sqrt{2}}$. Also from (36) it follows that the f^λ -induced metric has conformal parameter u for every $\lambda \in \mathbb{C} - \{0\}$, thus all the surfaces in the extended family have the same induced metric. Using formula (35) we obtain the Hopf quadratic differential of f^λ :

$$(88) \quad q_\lambda = \lambda^{-2} |\lambda|^2 dz^2.$$

Thus since $\delta_\lambda = h_\lambda q_\lambda = \lambda^{-2} \delta = \lambda^{-2} h dz^2$, then the δ -differential of f^λ is given by $\delta_\lambda = \lambda^{-2} |\lambda|^2 (\frac{h}{|\lambda|^2}) dz^2$. Thus the marginally trapped torus f^λ has mean curvature function $h_\lambda = \frac{h}{|\lambda|^2}$ which is a non-zero constant since λ does not depend on z . Hence f^λ has non-zero parallel mean curvature vector and so its null Gauss map G^λ is constrained Willmore. In the new (rotated) coordinate $w := \frac{|\lambda|}{\lambda} z$, κ_λ is real respect to w since $\kappa_\lambda dz^2 = \kappa dw^2$ and $\delta_\lambda = \frac{h}{|\lambda|^2} dw^2$, hence G^λ is isothermic for every $\lambda \in \mathbb{C} - \{0\}$.

Note that for $t = 2h(\frac{1}{|\lambda|^2} - 1)$ we recover the Calapso-Bianchi transformation f^t of the marginally trapped torus f .

The structure equations of f^λ in the extended frame $F^\lambda = (f^\lambda, f_z^\lambda, f_{\bar{z}}^\lambda, N_1^\lambda, N_2^\lambda)$, $\lambda \in \mathbb{C} - \{0\}$, are thus given by

$$(89) \quad \begin{aligned} f_{zz}^\lambda &= 2u_z f_z^\lambda + \lambda^{-2} |\lambda|^2 N_1^\lambda, \\ f_{\bar{z}\bar{z}}^\lambda &= -|\lambda|^4 e^{2u} f^\lambda + |\lambda|^4 e^{2u} \frac{h}{|\lambda|^2} (N_1^\lambda + N_2^\lambda), \\ \partial_z N_1^\lambda &= -\frac{h}{|\lambda|^2} f_z^\lambda - |\lambda|^{-4} e^{-2u} f_{\bar{z}}^\lambda, \\ \partial_z N_2^\lambda &= \frac{h}{|\lambda|^2} f_z^\lambda, \end{aligned}$$

REFERENCES

- [1] J.A. Aledo, J.A. Galvez and P. Mira, *Marginally trapped surfaces in L^4 and an extended Weierstrass-Bryant representation*, arXiv:math/0503702v1 [math.DG].
- [2] L.J. Alías and B. Palmer, *Conformal geometry of surfaces in lorentzian space forms*. Geometriae Dedicata 60, 301-315, 1996.
- [3] H. Anciaux, *Marginally trapped submanifolds in space forms with arbitrary signature* arXiv:1309.3875v3 [math.DG].
- [4] H. Anciaux and Y. Godoy, *Marginally trapped submanifolds in lorentzian space forms and in the lorentzian product of a space form by the real line*, arXiv:math/1301.4638v3 [math.DG].
- [5] A. I. Bobenko, *All constant mean curvature tori in R^3 , S^3 , H^3 in terms of theta-functions*, Math. Ann. 290 (1991), no. 2, 209245.
- [6] C. Bohle, G. P. Peters and U. Pinkall, *Constrained Willmore surfaces*, Calc. Var. Partial Differential Equations 32 (2008), 263-277. arXiv:math/0411479v3 [math.DG].

- [7] W. Blaschke, *Vorlesungen ueber Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitaetstheorie*, B. 3, bearbeitet von G. Thomsen, J. Springer, Berlin, 1929.
- [8] R. Bryant, *A Duality Theorem for Willmore surfaces* Journal of Differential Geometry 20 (1984), 23-53.
- [9] F. Burstall and J. Rawnsley, *Twistor theory for Riemannian symmetric spaces*, LNM Vol. 1424, Springer Verlag, 1990.
- [10] F. E. Burstall, F. Pedit, *Dressing orbits of harmonic maps*, Duke Math. J. 80 (1995) 353-382.
- [11] F.E. Burstall, F. Pedit, U. Pinkall, *Schwarzian derivatives and flows of surfaces*, Contemporary Mathematics 308, 3961, Amer. Math. Soc., Providence, RI, 2002.
- [12] D. M. J. Calderbank, *Moebius structures and two dimensional Einstein-Weyl geometry*. J. reine angew. Math. 504 (1998), 37-53.
- [13] B. Y. Chen and J. Van der Veken, *Classification of marginally trapped surfaces with parallel mean curvature vector in Lorentzian space forms*. Houston J. Math. 01/2010; 36(2):421-449.
- [14] B.Y. Chen, *Black holes, marginally trapped surfaces and quasi-minimal surfaces*, Tamkang Journal of Mathematics, Volume 40, Number 4, 313-341, Winter 2009.
- [15] J. L. Cabrerizo, M. Fernández and J.S. Gómez, *Isotropy and marginally trapped surfaces in a spacetime*. Class. Quantum Grav. 27 (2010) 135005 (12pp).
- [16] J. Eells and L. Lemaire, *Selected topics in harmonic maps*. CBMS Regional Conference Series in Mathematics, AMS 1983.
- [17] Norio Ejiri, *Willmore surfaces with a duality in $S^N(1)$* . Proc. London Math. Soc. (3) 57(1988), no. 2, 383-416.
- [18] Rahim Elghanmi, *Spacelike surfaces in Lorentzian manifolds*. Differential Geometry and its Applications 6 (1996) 199-218 North-Holland.
- [19] D. Ferus, F. Pedit, *Isometric immersions of space forms and soliton theory*. Math. Annalen 305.2 (1996): 329-342.
- [20] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Vol. 34 Graduate Studies in Mathematics, AMS.
- [21] H. Hertrich Jeromin, *Introduction to Moebius Differential Geometry* London Mathematical Society Lecture Note Series 300.
- [22] E. Hulett, *Superconformal harmonic surfaces in de Sitter space-times*. J. Geom. Phys., 55(2):179-206, 2005.
- [23] Huili Liu, *Weierstrass Type Representation for Marginally Trapped Surfaces in Minkowski 4-Space*, Math Phys Anal Geom (2013) 16:171178 DOI 10.1007/s11040-012-9125-7.
- [24] B. Palmer, *The conformal Gauss map and the stability of Willmore surfaces*, Ann. Global Anal. Geom. 9(1991), no. 3, 305-317.
- [25] B. Palmer, personal communication.
- [26] P. Wang, *Generalized polar transforms of spacelike isothermic surfaces*, arXiv:1111.1115 [math.DG].
- [27] J. Richter, *Conformal maps of a Riemannian surface onto the space of quaternions*, PhD thesis, TU-Berlin, 1997.
- [28] E. Ruh, and J. Vilms, *The tension field of the Gauss map*, Trans. Amer. Math. Soc, 149(1970), 569-573.
- [29] M. Spivak, *A Comprehensive Introduction to Differential Geometry* Vol. IV. Publish or perish 3rd edition 1999.
- [30] P. Wang, *Generalized polar transforms of spacelike isothermic surfaces*, arXiv:1111.1115v1 [math.DG].
- [31] J.C. Wood, *Harmonic Maps and Integrable Systems*, Aspects of Mathematics Volume E 23, A. Fordy, J.C. Wood editors 1994, pp 29-55.
- [32] Xiang Ma, *Willmore surfaces in S^n , Transforms and vanishing theorems*, Ph.D. Thesis, TU-Berlin 2005.

C.I.E.M. - FA.M.A.F. UNIVERSIDAD NACIONAL DE CÓRDOBA, CIUDAD UNIVERSITARIA, 5000 CÓRDOBA, ARGENTINA. PHONE/FAX: +54 351 4334052/51

E-mail address: hulett@famaf.unc.edu.ar